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THE BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ELLIPTIC OPERATORS SATISFYING A CARLESON CONDITION

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ABSTRACT. Let Ω be a Lipschitz domain in $\mathbb{R}^n$ $n \geq 2$, and $L = \mathrm{div} A\nabla$ be a second order elliptic operator in divergence form. We establish solvability of the Dirichlet regularity problem with boundary data in $H^{1,p}(\partial\Omega)$ and of the Neumann problem with $L^p(\partial\Omega)$ data for the operator L on Lipschitz domains with small Lipschitz constant. We allow the coefficients of the operator L to be rough obeying a certain Carleson condition with small norm. These results complete the results of [5] where $L^p(\partial\Omega)$ Dirichlet problem was considered under the same assumptions and [6] where the regularity and Neumann problems were considered on two dimensional domains.

1. Introduction

This paper continues the study, began in [5], of boundary value problem for second order divergence form elliptic operators, when the coefficients satisfy a certain natural, minimal smoothness condition. Specifically, we consider operators $L = \mathrm{div}(A\nabla)$ such that $A(X) = (a_{ij}(X))$ is strongly elliptic in the sense that there exists a positive constant Λ such that

$$\Lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(X)\xi_i\xi_j < \Lambda^{-1}|\xi|^2,$$

for all X and all $\vec{\xi} \in \mathbb{R}^n$. We do not assume symmetry of the matrix A. There are a variety of reasons for studying the non-symmetric situation. These include the connections with non-divergence form equations, and the broader issue of obtaining estimates on elliptic measure in the absence of special L^2 identities which relate tangential and normal derivatives.

In [14], the study of nonsymmetric divergence form operators with bounded measurable coefficients was initiated. In [11], the methods of [14] were used to prove A_∞ results for the elliptic measure of operators satisfying (a variant of) the Carleson measure condition.

Our main result is that under the assumption that

$$\delta(X)^{-1}\left(\mathrm{osc}_{B(X,\delta(X)/2)} a_{ij}\right)^2 \tag{1.1}$$

is the density of Carleson measure with small Carleson norm then the Dirichlet problem for operator L with boundary data in $H^{1,p}(\partial\Omega)$ is solvable. We also obtain the same result for the Neumann boundary value problem with $L^p(\partial\Omega)$ data.

Let us recall that the paper [5] considered the $L^p(\partial\Omega)$ Dirichlet problem under the same assumptions on the coefficients. It turns out that the regularity and Neumann boundary value problems are considerable more difficult and the progress on these problems under assumption (1.1) has been fairly slow. Only recently these two problems have been resolved on two dimensional domains in [6]. However, the proof relies fundamentally on the fact that the domain is two dimensional. In particular we are not currently able to generalise the concept of a conjugate solution to higher dimensional cases in our context. We should note, however, that such generalisations have been carried out in other contexts (see Theorem 9.3 of [1] and Section 1 of [16] which puts the result in context). Our paper relies on several recent advances, in particular [3] and [4] where a better understanding of the Dirichlet and regularity boundary values problems was obtained including the "duality" between the solvability of the Dirichlet boundary value problem and the regularity problem for the adjoint operator. In particular, this allows us to focus on solvability of the regularity problem for $p = 2$ and obtain solvability for values of p with no additional effort. We are able to avoid the use of the p-adapted square function introduced in [5] and which was essential in establishing solvability of the L^p Dirichlet problem. The solvability of the Neumann problem is trickier as there is no appropriate analogue of the results in [3]. We overcome this by using the solvability of the regularity problem, which we established here, and induction (see Lemma 6.2).

Operators whose coefficients satisfy small or vanishing Carleson condition (1.1) arise in several contexts. For example, consider the boundary value problems associated with a smooth elliptic operator in the region above a graph $t = \varphi(x)$. When φ is C^1, it was shown in [7] that the Dirichlet, regularity and Neumann boundary value problems are solvable with data in L^p for $1 < p < \infty$, by the method of layer potentials. Our main theorem and the result of [5] show that the Dirichlet and regularity problems are solvable in this range of p when the boundary of the domain is defined by $t = \varphi(x)$ where $\nabla\varphi \in L^\infty \bigcap VMO$. The Carleson condition arises naturally here; the function $v = u \circ \Phi$ for the mapping $\Phi : \mathbb{R}^n_+ \to \{X = (x,t); t > \phi(x)\}$ defined by (4.10) solves an elliptic equation in $\mathbb{R}^n_+$ with coefficients satisfying (1.1).

The paper is organized as follows. In Section 2, we give definitions and state our main results. Here we reduce the proofs of these main results to the case $p = 2$ and integer p for the regularity and Neumann problems, respectively. The proofs are applications of several general results, including the recent advances mentioned above. Section 3 considers in detail the square function of the gradient of a solution. Section 4 explores the comparability of the square and non-tangential maximal functions and in Section 5 the $p = 2$ version of our regularity result is established. In Section 6 we revisit bounds on the square function, this time by the co-normal derivative. Finally Section 7 establishes a version of our Neumann result for integer values of p.

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2. Definitions and Statements of Main Theorems

Let us begin by introducing Carleson measures and the square function on domains which are locally given by the graph of a function. We shall assume that our domains are Lipschitz.

Definition 2.1. *$\mathbb{Z} \subset \mathbb{R}^n$ is an L-cylinder of diameter d if there exists a coordinate system (x,t) such that*

$$\mathbb{Z} = \{(x,t)\ :\ |x| \le d,\ -2Ld \le t \le 2Ld\}$$

and for $s > 0$,

$$s\mathbb{Z} := \{(x,t)\ :\ |x| < sd, -2Ld \le t \le 2Ld\}.$$

Definition 2.2. *$\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with Lipschitz 'character' (L, N, C_0) if there exists a positive scale r_0 and at most N L-cylinders $\{\mathbb{Z}_j\}_{j=1}^N$ of diameter d, with $\frac{r_0}{C_0} \le d \le C_0 r_0$ such that*

(i) $8\mathbb{Z}_j \cap \partial\Omega$ is the graph of a Lipschitz function ϕ_j, $\|\phi_j\|_\infty \le L,\ \phi_j(0) = 0$,

(ii) $\partial\Omega = \bigcup_j (\mathbb{Z}_j \cap \partial\Omega)$,

(iii) $\mathbb{Z}_j \cap \Omega \supset \left\{(x,t) \in \Omega\ :\ |x| < d,\ \mathrm{dist}\,((x,t), \partial\Omega) \le \dfrac{d}{2}\right\}$.

Definition 2.3. *Let Ω be a Lipschitz domain. For $Q \in \partial\Omega$, $X \in \Omega$ and $r > 0$ we write:*

$$\Delta_r(Q) = \partial\Omega \cap B_r(Q), \qquad T(\Delta_r) = \Omega \cap B_R(Q),$$
$$\delta(X) = dist(X, \partial\Omega).$$

Definition 2.4. *Let $T(\Delta_r)$ be the Carleson region associated to a surface ball Δ_r in $\partial\Omega$, as defined above. A measure μ in Ω is Carleson if there exists a constant $C = C(r_0)$ such that for all $r \le r_0$,*

$$\mu(T(\Delta_r)) \le C\sigma(\Delta_r).$$

The best possible C is the Carleson norm. When $d\mu$ is Carleson we write $d\mu \in \mathcal{C}$.
If $\lim_{r_0 \to 0} C(r_0) = 0$, then we say that the measure μ satisfies the vanishing Carleson condition, and we denote this by writing $d\mu \in \mathcal{C}_V$.

Definition 2.5. *A cone of aperture a is a non-tangential approach region for $Q \in \partial\Omega$ of the form*

$$\Gamma_a(Q) = \{X \in \Omega : |X - Q| \le (1+a)\ \ dist(X, \partial\Omega)\}.$$

Sometimes it will be necessary to truncate $\Gamma_a(Q)$, so we define $\Gamma_{a,h}(Q) = \Gamma_a(Q) \cap B_h(Q)$.

Definition 2.6. *If $\Omega \subset \mathbb{R}^n$, the square function of a function u defined on Ω, relative to the family of cones $\{\Gamma_a(Q)\}_{Q\in\partial\Omega}$, is*

$$S_{[a]}(u)(Q) = \left(\int_{\Gamma_a(Q)} |\nabla u(X)|^2(X) dist(X, \partial\Omega)^{2-n} dX \right)^{1/2}$$

at each $Q \in \partial\Omega$. The non-tangential maximal function relative to $\{\Gamma_a(Q)\}_{Q\in\partial\Omega}$ is

$$N_a(u)(Q) = \sup_{X\in\Gamma_a(Q)} |u(X)|$$

at each $Q \in \partial\Omega$ The truncation at height h of the non-tangential maximal function is defined by $N_{[a],h}(u)(Q) = \sup_{X\in\Gamma_a(Q)\cap B_h(Q)} |u(X)|$, with a similar notation $S_{[a],h}$ for truncated square function.
It will often be convenient to supress one or both of the parameters a and h in the square and non-tangential functions when their values do not play a significant role in an argument. So we may write S, $S_{[a]}$ or S_h to denote $S_{[a],h}$ when no confusion should arise. Similarly we may abreviate $N_{[a],h}$ as N, $N_{[a]}$ or N_h.
We also define the following variant of the non-tangential maximal function:

$$\widetilde{N}(u)(Q) = \widetilde{N}_{[a],h}(u)(Q) = \sup_{X\in\Gamma_{a,h}(Q)} \left(\fint_{B_{\delta(X)/2}(X)} |u(Y)|^2\, dY \right)^{\frac{1}{2}}. \tag{2.1}$$

Definition 2.7. *Let $1 < p \leq \infty$. The Dirichlet problem with data in $L^p(\partial\Omega, d\sigma)$ is solvable (abbreviated $(D)_p$) if for every $f \in C(\partial\Omega)$ the weak solution u to the problem $Lu = 0$ with continuous boundary data f satisfies the estimate*

$$\|N(u)\|_{L^p(\partial\Omega,d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega,d\sigma)}.$$

The implied constant depends only the operator L, p, and the Lipschitz character of the domain as measured by the triple (L, N, C_0) of Definition 2.2.

Definition 2.8. *Let $1 < p < \infty$. The regularity problem with boundary data in $H^{1,p}(\partial\Omega)$ is solvable (abbreviated $(R)_p$), if for every $f \in H^{1,p}(\partial\Omega) \cap C(\partial\Omega)$ the weak solution u to the problem*

$$\begin{cases} Lu &= 0 \quad \text{in } \Omega \\ u|_{\partial B} &= f \quad \text{on } \partial\Omega \end{cases}$$

satisfies

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_T f\|_{L^p(\partial\Omega)}.$$

Again, the implied constant depends only the operator L, p, and the Lipschitz character of the domain.

Definition 2.9. *Let $1 < p < \infty$. The Neumann problem with boundary data in $L^p(\partial\Omega)$ is solvable (abbreviated $(N)_p$), if for every $f \in L^p(\partial\Omega) \cap C(\partial\Omega)$ such that $\int_{\partial\Omega} f d\sigma = 0$ the weak solution u to the problem*

$$\begin{cases} Lu &= 0 \quad \text{in } \Omega \\ A\nabla u \cdot \nu &= f \quad \text{on } \partial\Omega \end{cases}$$

satisfies

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

Again, the implied constant depends only the operator L, p, and the Lipschitz character of the domain. Here ν is the outer normal to the boundary $\partial\Omega$.

We are now ready to formulate our main results.

Theorem 2.10. *Let $1 < p < \infty$ and let $\Omega \subset M$ be a Lipschitz domain with Lipschitz norm L on a smooth Riemannian manifold M and $Lu = div(A\nabla u)$ be an elliptic differential operator defined on Ω with ellipticity constant Λ and coefficients which are such that*

$$\delta(X)^{-1}\left(osc_{B(X,\delta(X)/2)}a_{ij}\right)^2 \tag{2.2}$$

is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(R)_p$ regularity problem

$$\begin{cases} Lu & = 0 \quad in\ \Omega \\ u|_{\partial\Omega} & = f \quad on\ \partial\Omega \\ \widetilde{N}(\nabla u) & \in L^p(\partial\Omega) \end{cases}$$

is solvable for all f with $\|\nabla_T f\|_{L^p(\partial\Omega)} < \infty$. Moreover, there exists a constant $C = C(\Lambda, n, a, p) > 0$ such that

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \le C\|\nabla_T f\|_{L^p(\partial\Omega)}. \tag{2.3}$$

In particular, if the domain Ω is C^1 and $A = (a_{ij})$ satisfies the vanishing Carleson condition, then the regularity problem is solvable for all $1 < p < \infty$. More generally, if the boundary of the domain Ω is given locally by a function ϕ such that $\nabla\phi$ belongs to $L^\infty \cap VMO$, then, once again, the regularity problem is solvable for all $1 < p < \infty$.

Proof. As follows from Theorem 5.2, the $(R)_2$ regularity problem is solvable for operators satisfying (2.2), provided ε is sufficiently small. To complete the proof we will use [3, Theorem 1.1]. (There is also an older result by [17] for symmetric operators which should be adaptable to the non-symmetric case.)

According to [3, Theorem 1.1] $(R)_2$ solvability implies the solvability of $(R)_{HS^1}$ (this is an end-point Hardy-Sobolev space boundary-value problem corresponding to $p = 1$). We also have from [3, Theorem 1.1] that under the assumption $(R)_{HS^1}$ is solvable we have for $p \in (1, \infty)$:

$$(R)_p \text{ is solvable if and only if } (D^*)_{p'} \text{ is solvable for } p' = p/(p-1). \tag{2.4}$$

Here $(D^*)_{p'}$ denotes the $L^{p'}$ Dirichlet problem for the adjoint operator $L^*u = \mathrm{div}(A^t\nabla u)$.

However by [5, Corollary 2.3] the $L^{p'}$ Dirichlet problem for the operator L^* is solvable under the assumptions of Theorem 2.10 (for sufficiently small $\varepsilon = \varepsilon(p') > 0$). Hence by (2.4) the $(R)_p$ problem for the operator L is solvable proving our claim. □

As follows from the proof given above we also have a result for the endpoint $p = 1$.

Corollary 2.11. *Under the same assumptions as in Theorem 2.10 the $(R)_{HS^1}$ regularity problem for the operator L is solvable for all f with $\nabla_T f$ in the atomic Hardy space (c.f.* [3, Theorem 2.3]*). Moreover, there exists a constant $C = C(\Lambda, n, a) > 0$ such that*

$$\|\widetilde{N}(\nabla u)\|_{L^1(\partial\Omega)} \le C\|\nabla_T f\|_{\hbar^1(\partial\Omega)}. \tag{2.5}$$

For the Neumann problem we have the following:

Theorem 2.12. *Let $1 < p < \infty$ and let $\Omega \subset M$ be a Lipschitz domain with Lipschitz norm L on a smooth Riemannian manifold M and $Lu = div(A\nabla u)$ be an elliptic differential operator defined on Ω with ellipticity constant Λ and coefficients which are such that*

$$\delta(X)^{-1}\left(osc_{B(X,\delta(X)/2)}a_{ij}\right)^2 \tag{2.6}$$

is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the $(N)_p$ Neumann problem

$$\begin{cases} Lu & = 0 \quad \text{in } \Omega \\ A\nabla u \cdot \nu & = f \quad \text{on } \partial\Omega \\ \widetilde{N}(\nabla u) & \in L^p(\partial\Omega) \end{cases}$$

is solvable for all f in $L^p(\partial\Omega)$ such that $\int_{\partial\Omega} f d\sigma = 0$. Moreover, there exists a constant $C = C(\Lambda, n, a, p) > 0$ such that

$$\|\widetilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}. \tag{2.7}$$

In particular, if the domain Ω is C^1 and $A = (a_{ij})$ satisfies the vanishing Carleson condition, then the Neumann problem is solvable for all $1 < p < \infty$. More generally, if the boundary of the domain Ω is given locally by a function ϕ such that $\nabla\phi$ belongs to $L^\infty \cap VMO$, then, once again, the Neumann problem is solvable for all $1 < p < \infty$.

Proof. As follows from Theorem 7.1 the $(N)_p$ Neumann problem is solvable for operators satisfying (7.1), provided ε is sufficiently small and p is an integer. To replace the condition (7.1) by (2.6) we use the same idea as [5, Corollary 2.3] and Theorem 5.2. For a matrix A satisfying (7.1) with ellipticity constant Λ one can find (by mollifying the coefficients of A) a new matrix $\widetilde{A}$ with same ellipticity constant Λ such that $\widetilde{A}$ satisfies (7.1) and

$$\sup\{\delta(X)^{-1}|(A - \widetilde{A})(Y)|^2; Y \in B(X, \delta(X)/2)\} \tag{2.8}$$

is a Carleson norm. Moreover, if the Carleson norm for matrix A is small (on balls of radius $\leq r_0$), so are the Carleson norms of (7.1) for $\widetilde{A}$ and (2.8). Hence by Theorem 7.1 the $(N)_p$ regularity problem is solvable for the operator $\widetilde{L}u = \operatorname{div}(\widetilde{A}\nabla u)$.

The solvability of the Neummann problem for perturbed operators satisfying (2.8) has been studied in [13]. It follows by [13, Theorem 2.2] that the L^p Neumann problem for the operator L is solvable, provided (2.8) has small Carleson norm and the regularity $(R)_p$ and Neumann $(N)_p$ problems are solvable for $\widetilde{L}$. Actually, the results in [13] are stated for symmetric operators, however careful study of the proof of [13, Theorem 2.2] reveals that symmetry is not necessary.

However by Theorem 2.10 the $(R)_p$ regularity problem for $\widetilde{L}$ is solvable provided the Carleson norm of (7.1) is sufficiently small and $(N)_p$ Neumann problem for $\widetilde{L}$ is solvable by Theorem 7.1. Hence we have solvability of the Neumann problem $(N)_p$ for L by [13, Theorem 2.2].

If $p > 1$ is not an integer we use [12, Theorem 6.2]. (This result is also stated for symmetric operators, however, once again, symmetry is not necessary.) This theorem implies that $(N)_p$ is solvable, provided $(R)_k$ and $(N)_k$ are solvable, where k is any integer larger than p. $\square$

3. The Square Function for the Gradient of a Solution

In this section we shall assume that Ω is a smooth domain on a smooth compact Riemannian manifold M. As we shall see the case of a Lipschitz domain with small coefficients can be reduced to this situation via a pull back map of Dahlberg, Nečas and Stein (see (4.10)).

We note that this case includes the most usual situation when $\Omega \subset \mathbb{R}^n$ is a bounded domain as in this case we can think of Ω as being embedded into a large torus $\mathbb{T}^n$. We aim to establish local results near $\partial\Omega$. For this reason we introduce a convenient localization and parametrization of points near $\partial\Omega$.

We want to write any point $X \in \Omega$ near $\partial\Omega$ as $X = (x,t)$ where $x \in \partial\Omega$ and $t > 0$. The boundary $\partial\Omega$ itself then will be the set $\{(y,0); y \in \partial\Omega\}$. One way to get such a parametrization is to consider the inner normal N to the boundary $\partial\Omega$. The assumption that $\partial\Omega$ is smooth implies smoothness of N. On Ω we have a smooth underlying metric of the manifold M.

We consider the geodesic flow $\mathcal{F}_t$ in this metric starting at any point $x \in \partial\Omega$ in the direction $N(x)$. We assign to a point $X \in \Omega$ coordinates (x,t) if $X = \mathcal{F}_t x$. This means that starting at $x \in \partial\Omega$ it takes time t for the flow to get to the point X. It's an easy exercise that the map $(x,t) \mapsto X = \mathcal{F}_t x$ is a smooth diffeomorphism for small $t \leq t_0$. Using this parametrization we consider the set $\Omega_{t_0} = \{(x,t); (x,0) \in \partial\Omega \text{ and } 0 < t < t_0\}$.

Let us now deal with the issue of the metric. We want to work with the simplest possible metric on Ω available. Since we only work on Ω_{t_0} we take our metric tensor there to be a product $d\sigma \otimes dt$ where $d\sigma$ is the original metric tensor on Ω restricted to $\partial\Omega$. The product metric $d\sigma \otimes dt$ is different from the original metric on Ω, but they are both smooth and comparable, that is the distances between points are comparable. Now we express the operator L in this new product metric.

We note that under this pullback the new coefficients of our operator are going to satisfy the same Carleson condition as the original coefficients with Carleson norm comparable to the original. We observe in particular that the Carleson condition implies that $\nabla A \in L^\infty_{loc}(\Omega_{t_0})$ hence any solution of $Lu = 0$ on Ω_{t_0} has a well defined pointwise gradient ∇u. Furthermore, in the product metric $d\sigma \otimes dt$, the gradient ∇u can be written as

$$\nabla u = (\nabla_T u, \partial_t u),$$

where ∇_T is the gradient restricted to the $n-1$ dimensional set $\partial\Omega \times \{t = \text{const}\}$.

Frequently throughout the paper it will be useful to localize to a single coordinate patch. The following definition gives a precise notion of coordinate frame.

Definition 3.1. *Let $\partial\Omega$ be a smooth $n-1$ dimensional compact Riemannian manifold. We say that a finite collection of smooth vector fields $(\vec{T}_\tau)_{\tau=1}^m$ $(m \geq n-1)$ is an coordinate frame for $\partial\Omega$ if:*

- *there is a finite collection of open sets* $U_1, U_2, \dots, U_k$ *in* $\mathbb{R}^{n-1}$ *and smooth diffeomorphisms* $\varphi_\ell : U_\ell \to \partial\Omega$ *such that* $\bigcup_\ell \varphi_\ell(\widetilde{U}_\ell)$ *covers* $\partial\Omega$*, where* $\widetilde{U}_\ell$ *is an open subset of* U_ℓ *such that* $\overline{\widetilde{U}_\ell} \subset U_\ell$*;*
- *for each* $1 \le \ell \le k$ *there exist a set* $A_\ell \subset \{1, 2, \dots, m\}$ *such that* $|A_\ell| = n-1$ *and*
$$\{\varphi_\ell^*(\vec{T}_\tau)\big|_{\widetilde{U}_\ell};\ \tau \in A_\ell\} = \{\tfrac{\partial}{\partial x_j}\big|_{\widetilde{U}_\ell};\ j = 1, 2, \dots, n-1\}.$$
That is the pullback of the vectors $\vec{T}_\tau$ *to* U_ℓ*,* $\tau \in A_\ell$ *restricted to* $\widetilde{U}_\ell$ *are just coordinate vector fields on* $\widetilde{U}_\ell$*.*

Clearly, $\partial\Omega$ has at least one such coordinate frame. Indeed, the existence of a finite collection $(U_\ell, \widetilde{U}_\ell, \varphi_\ell)$ satisfying all assumptions of the previous definition follows from the fact that $\partial\Omega$ is a smooth compact Riemannian manifold. Then on each U_ℓ we consider vector fields $\psi_\ell \frac{\partial}{\partial_j}$, $j = 1, 2, \dots, n-1$ where $\psi_\ell \in C_0^\infty(U_\ell)$ is a smooth cutoff function such that $\psi_\ell\big|_{\widetilde{U}_\ell} = 1$ and $0 \le \psi_\ell \le 1$ on U_ℓ. Then
$$\{\varphi_{\ell *}(\psi_\ell \tfrac{\partial}{\partial_j}); 1 \le \ell \le k,\ 1 \le j \le n-1\}$$
is one such coordinate frame. Here $\varphi_{\ell *}$ denotes the push-forward of a vector field from U_ℓ onto $\partial\Omega$.

We start with the following key lemma for the square function $S(\nabla_T u)$.

Lemma 3.2. *Let* u *be a solution to* $Lu = \mathrm{div} A \nabla u = 0$*, where* L *is an elliptic differential operator defined on* Ω_{t_0} *with bounded coefficients such that*
$$\sup\{\delta(X)|\nabla a_{ij}(Y)|^2\ :\ Y \in B_{\delta(X)/2}(X)\} \tag{3.1}$$
is the density of a Carleson measure on all Carleson boxes of size at most r_0 *with norm* $C(r_0)$*.*

Then there exists $r_1 > 0$ *and* $K > 0$ *depending only on the geometry of the domain* Ω*, elliptic constant* Λ *and dimension* n *such that*
$$\begin{aligned}\int_{\partial\Omega} S^2_{r/2}(\nabla_T u)\, d\sigma &\simeq \iint_{\partial\Omega\times(0,r/2)} |\nabla(\nabla_T u(X))|^2 \delta(X)\, dX \\ &\le\ K \int_{\partial\Omega} |\nabla_T u|^2\, d\sigma + \varepsilon \int_{\partial\Omega} N_r^2(\nabla u) d\sigma + \frac{K}{r}\|\nabla u\|^2_{L^2(\Omega)},\end{aligned} \tag{3.2}$$
for all $r \le \min\{r_0, r_1, t_0\}$*. Here* $\varepsilon > 0$ *depends on the Carleson norm* $C(r_0)$ *and* $\varepsilon \to 0+$ *if* $C(r_0) \to 0$*.* N_r *denotes is the non-tangential maximal function truncated at height* r*,* $\delta(X) = t$ *for a point* $X = (x, t) \in \partial\Omega \times (0, t_0)$*,* $\nabla_T u$ *is the tangential gradient of* u *on* $\partial\Omega \times \{t\}$ *and* dX *is the product measure* $d\sigma dt$*.*

Proof. In order to establish (3.2) we localize to coordinates. Let $U = U_\ell$ be one of the sets from Definition 3.1 with corresponding map $\varphi = \varphi_\ell$, equally set $\widetilde{U} = \widetilde{U}_\ell$. We can now consider the operator L as being defined on an open subset $U \times (0, t_0)$ of $\mathbb{R}^n_+$, where $\partial\Omega$ corresponds to the hyperplane $\{(x, 0); x \in U\}$. We achieve this by pulling back the coefficients of L from Ω_{t_0} to $U \times (0, t_0)$ using the smooth map $\Phi : (x, t) \mapsto (\varphi(x), t)$. At this stage we also pull back the product metric $d\sigma \otimes dt$ from $\partial\Omega \times (0, t_0)$ to $U \times (0, t_0)$

and we get another product metric that we (in a slight abuse of notation) still denote by $d\sigma \otimes dt$ on $U \times (0, t_0)$.

Since we are going to use a partition of unity we also consider a smooth cutoff function $\phi(x,t) = \phi(x)$ defined on $\mathbb{R}^n_+$, independent of the $t = x_n$ variable such that

$$0 \le \phi(x,t) \le 1, \quad \operatorname{supp} \phi \subset \widetilde{U} \times \mathbb{R}.$$

Instead of the left-hand side of (3.2) (by the ellipticity of the coefficients) we are going to estimate a similar object

$$\iint_{U\times(0,r)} \frac{a_{ij}}{a_{nn}} (\partial_i V)(\partial_j V) \phi t \, d\sigma \, dt, \tag{3.3}$$

for functions $V = \nabla u \cdot \vec{T}_\tau$, for $1 \le \tau \le m$. Here and below we use the summation convention and consider the variable t to be the n-th variable. We begin by integrating by parts

$$\begin{aligned}\iint_{U\times(0,r)} \frac{a_{ij}}{a_{nn}} (\partial_i V)(\partial_j V) \phi t \, d\sigma \, dt &= \frac{1}{2} \int_{U\times\{r\}} \partial_j (|V|^2) \frac{a_{ij}}{a_{nn}} \phi t \nu_i \, d\sigma - \\ - \iint_{U\times(0,r)} \frac{1}{a_{nn}} V(LV) \phi t \, d\sigma \, dt &- \iint_{U\times(0,r)} V(\partial_j V) a_{ij} \partial_i \left(\frac{\phi t}{a_{nn}} \right) d\sigma \, dt.\end{aligned} \tag{3.4}$$

Here ν_i is the i-th component of the outer normal ν, which (given we consider a product metric) is is just the vector e_n for the boundary $U \times \{r\}$. Hence the first term is non-vanishing only for $i = n$. We work on the last term, as it is the most complicated. This one splits into three new terms, one when the derivative hits t (where only the term with $i = n$ will remain) and another two when it hits ϕ and $1/a_{nn}$:

$$\begin{aligned}- \iint_{U\times(0,r)} V(\partial_j V) \frac{a_{nj}}{a_{nn}} \phi \, d\sigma \, dt \; &- \iint_{U\times(0,r)} V(\partial_j V) \frac{a_{ij}}{a_{nn}} (\partial_i \phi) t \, d\sigma \, dt \\ &+ \iint_{U\times(0,r)} V(\partial_j V) \frac{a_{ij}}{a_{nn}^2} (\partial_i a_{nn}) \phi t \, d\sigma \, dt.\end{aligned} \tag{3.5}$$

Consider now the first term of (3.5). For $j = n$, as ϕ is independent of $x_n = t$, we only get

$$-\frac{1}{2} \iint_{U\times(0,r)} \partial_n (|V|^2 \phi) \, d\sigma \, dt = \frac{1}{2} \int_U |V|^2 \phi \, d\sigma - \frac{1}{2} \int_{U\times\{r\}} |V|^2 \phi \, d\sigma \tag{3.6}$$

For $j < n$ the first term of (3.5) is handled as follows. We introduce an artificial term $1 = \partial_n t$ inside the integral and integrate by parts.

$$\begin{aligned} &- \frac{1}{2}\iint_{U\times(0,r)} \partial_j(|V|^2)\frac{a_{nj}}{a_{nn}}\phi(\partial_n t)\, d\sigma\, dt = -\frac{1}{2}\int_{U\times\{r\}} \partial_j(|V|^2)\frac{a_{nj}}{a_{nn}}\phi t\, d\sigma \\ &+ \frac{1}{2}\iint_{U\times(0,r)} \partial_n\left(\partial_j(|V|^2)\frac{a_{nj}}{a_{nn}}\phi\right) t\, d\sigma dt = -\frac{1}{2}\int_{U\times\{r\}} \partial_j(|V|^2)\frac{a_{nj}}{a_{nn}}\phi t\, d\sigma \\ &+ \frac{1}{2}\iint_{U\times(0,r)} \partial_j\partial_n(|V|^2)\frac{a_{nj}}{a_{nn}}\phi t\, d\sigma dt + \frac{1}{2}\iint_{U\times(0,r)} \partial_j(|V|^2)\partial_n\left(\frac{a_{nj}}{a_{nn}}\right)\phi t\, d\sigma dt. \end{aligned} \tag{3.7}$$

The first term here gets completely cancelled out by the first term of (3.4) as they have opposite signs. The second term can be further integrated by parts and we obtain

$$\begin{aligned} \frac{1}{2}\iint_{U\times(0,r)} \partial_j\partial_n(|V|^2)\frac{a_{nj}}{a_{nn}}\phi t\, d\sigma dt &= -\frac{1}{2}\iint_{U\times(0,r)} \partial_n(|V|^2)\partial_j\left(\frac{a_{nj}}{a_{nn}}\right)\phi t\, d\sigma dt \\ &- \frac{1}{2}\iint_{U\times(0,r)} \partial_n(|V|^2)\frac{a_{nj}}{a_{nn}}(\partial_j\phi)t\, d\sigma dt \end{aligned} \tag{3.8}$$

We now notice that the last term of (3.5), the third term on the righthand side of (3.7) and the first on the righthand side of (3.8) are of same type and are bounded from above by

$$C\iint_{U\times(0,r)} |V||\nabla V||\nabla A|\phi t\, d\sigma dt. \tag{3.9}$$

Here ∇A stands generically for either ∇a_{nj}, ∇a_{nn}. Estimating (3.9) further we see that, using the Cauchy-Schwarz inequality, (3.9) is less than

$$C\left(\iint_{U\times(0,r)} |V|^2|\nabla A|^2\phi t\, d\sigma dt\right)^{1/2}\left(\iint_{U\times(0,r)} |\nabla V|^2\phi t\, d\sigma dt\right)^{1/2}. \tag{3.10}$$

Using the Carleson condition on the coefficients, and the fact that the Carleson norm is less than ε we get that this can be further written as

$$\begin{aligned} &C\varepsilon^{1/2}\left(\int_U N_r(V)^2 dy\right)^{1/2}\left(\iint_{U\times(0,r)} |\nabla V|^2\phi t\, d\sigma dt\right)^{1/2} \\ &\leq \frac{\Lambda^2}{2}\iint_{U\times(0,r)} |\nabla V|^2\phi t\, d\sigma dt + \frac{C^2\varepsilon}{2}\int_U N_r(V)^2 dy, \end{aligned} \tag{3.11}$$

where the last line follows from the inequality between arithmetic and geometric means. We observe that the first term on the second line is no more than one half of (3.3) and hence can be incorporated there.

Let us summarize what we have. For some constant $C > 0$ and ε (the Carleson norm) we have that

$$\begin{aligned}(3.12)\qquad &\iint_{U\times(0,r)} \frac{a_{ij}}{a_{nn}}(\partial_i V)(\partial_j V)\phi t\,d\sigma\,dt\\ \leq\;& C\varepsilon\int_U N_r(V)^2dy+\int_U |V|^2\phi\,d\sigma-\int_{U\times\{r\}}|V|^2\phi\,d\sigma+\\ +\;&\int_{U\times\{r\}}\partial_n(|V|^2)\phi t\,d\sigma-\iint_{U\times(0,r)}\frac{1}{a_{nn}}V(LV)\phi t\,d\sigma\,dt+E.\end{aligned}$$

The fourth term on the righthand side is the first term of (3.4) for $i=j=n$. Here

$$E=-\iint_{U\times(0,r)}\partial_j(|V|^2)\frac{a_{ij}}{a_{nn}}(\partial_i\phi)t\,d\sigma\,dt-\iint_{U\times(0,r)}\partial_n(|V|^2)\frac{a_{nj}}{a_{nn}}(\partial_j\phi)t\,d\sigma dt. \tag{3.13}$$

We call E "the error terms" these are the second term of (3.5) and the second term on the righthand side of (3.8). Both terms are of same type and contain $\partial_i\phi$ for $i<n$. (Recall that $\partial_n\phi=0$).

At this point we have to use the fact that $V=\nabla u\cdot\vec{T}_i$, for $1\leq i\leq m$, where $(\vec{T}_i)_{i=1}^m$ is a frame from Definition 3.1. It follows that in our local coordinates

$$V=\sum_{k<n}b^k v_k,\qquad \text{for some smooth functions } b^k \text{ on } U.$$

Here

$$v_k=\partial_k u=\frac{\partial u}{\partial x_k},\qquad \text{for } k=1,2,\dots,n-1.$$

We denote by $v_n=\partial_t u$. We observe that each v_k is a solution of the following auxiliary inhomogeneous equation:

$$\mathrm{div}(A\nabla v_k)=Lv_k=-\mathrm{div}((\partial_k A)\mathbf{v})=\mathrm{div}\vec{F}_k, \tag{3.14}$$

where the i-th component of the vector $\vec{F}_k$ is $(\vec{F}_k)^i=-(\partial_k a_{ij})\partial_j u=-(\partial_k a_{ij})v_j$.

It remains to deal with the second term of the last line in (3.12). Clearly,

$$LV=\sum_{k<n}\left[\partial_i(a_{ij}(\partial_j b^k))v_k+a_{ij}(\partial_j b^k)\partial_i v_k+a_{ij}(\partial_i b^k)\partial_j v_k+b^k Lv_k\right]. \tag{3.15}$$

We will have to deal with these four terms. We start with the second and third ones as they are the easiest. We observe that since b^i are smooth, both b^i and ∇b^i actually satisfy the vanishing Carleson condition. Hence these two terms put into the expression

$$\iint_{U\times(0,r)}\frac{1}{a_{nn}}V(LV)\phi t\,d\sigma\,dt \tag{3.16}$$

can be estimated by

$$C\sum_{k<n}\iint_{U\times(0,r)}|\nabla u||\nabla v_k||B|\phi t\,d\sigma dt, \tag{3.17}$$

where B stands for a generic coefficient such as $a_{ij}(\partial_j b^k)$ or $a_{ij}(\partial_i b^k)$ with $|B|^2 t$ being a density of a vanishing Carleson measure. Hence in the same spirit as we dealt with (3.9) we get

$$\sum_{k<n}\iint_{U\times(0,r)}|\nabla u||\nabla v_k||B|\phi t\,d\sigma dt \leq \tag{3.18}$$

$$\leq \quad K\sum_{k<n}\iint_{U\times(0,r)}|\nabla v_k|^2\phi t\,d\sigma dt + C(K)\varepsilon\int_U N_r(\nabla u)^2 dy.$$

The K is this formula can be arbitrary small, ε only depends on the Carleson norm of $|B|^2 t$ near the boundary, hence it can also be arbitrary small by making $r_1 > 0$ smaller if necessary. We choose K sufficiently small so that the first term on the second line of (3.18) can be hidden on the left-hand side of (3.2).

Next we look at the first term of (3.15) as we place it into (3.16). We obtain

$$\left|\iint_{U\times(0,r)}\frac{1}{a_{nn}}V\partial_i(a_{ij}(\partial_j b^k))v_k\phi t\,d\sigma\,dt\right| \leq C\iint_{U\times(0,r)}|\nabla u|^2|\nabla B|\phi t\,d\sigma\,dt. \tag{3.19}$$

Here $|\nabla B|^2 t$ satisfies the Carleson condition with small norm since $|\nabla a_{ij}|^2 t$ has a small Carleson norm, and $\partial_j b^k$ and its higher derivatives are smooth functions (and so they satisfy the vanishing Carleson condition). Hence by the Cauchy-Schwarz

$$\iint_{U\times(0,r)}|\nabla u|^2|\nabla B|\phi t\,d\sigma\,dt \quad\leq\quad \left(\iint_{U\times(0,r)}|\nabla u|^2 t\,d\sigma dt\right)^{1/2}\left(\iint_{U\times(0,r)}|\nabla u|^2|\nabla B|^2 t\,d\sigma dt\right)^{1/2}$$

$$\leq \quad Cr\varepsilon^{1/2}\int_U N_r^2(\nabla u)d\sigma. \tag{3.20}$$

Here we observe that the last term on the first line is of the same type as the first term in (3.10) we have handled before. It follows that this term is small even if the Carleson condition has a large norm by choosing r small.

We now handle the last term of (3.15) using (3.14). Placing this into (3.16) yields a term

$$\sum_{k<n}\iint_{U\times(0,r)}\frac{1}{a_{nn}}Vb^k\partial_i((\partial_k a_{ij})v_j)\phi t\,d\sigma dt = \tag{3.21}$$

$$= \quad -\sum_{k<n}\iint_{U\times(0,r)}\partial_i\left(\frac{b^k\phi}{a_{nn}}Vt\right)(\partial_k a_{ij})v_j\,d\sigma dt + \int_{U\times\{r\}}\frac{b^k\partial_k a_{nj}}{a_{nn}}Vv_j\phi t\,d\sigma,$$

where we integrate by parts and only obtain a boundary term when $i = n$. We deal now with the solid integral. This gives

$$
\begin{aligned}
(3.22) \qquad & - \sum_{k<n} \iint_{U\times(0,r)} \partial_i\left(\frac{b^k}{a_{nn}}\right)(\partial_k a_{ij})V v_j \phi t\, d\sigma dt - \\
& - \sum_{k<n} \iint_{U\times(0,r)} \frac{b^k(\partial_k a_{ij})}{a_{nn}} \partial_i V v_j\, \phi t\, d\sigma dt - \\
& - \sum_{k<n} \iint_{U\times(0,r)} \frac{b^k\partial_k a_{ij}}{a_{nn}} V v_j (\partial_i\phi) t\, d\sigma dt\; - \\
& - \sum_{k<n} \iint_{U\times(0,r)} \frac{b^k\partial_k a_{nj}}{a_{nn}} V v_j\, \phi\, d\sigma dt,
\end{aligned}
$$

where the last term only appears for $i = n$ (as $\partial_n(t) = 1$). We notice that the first term here is of the same type as the first term in (3.10) and hence bounded by $\varepsilon\int_U N_r(\nabla u)^2\, d\sigma$. The second term is handled exactly as (3.9) (noticing that $|v_j| \le |\nabla u|$). Hence this term is (in absolute value) no greater than

$$K \iint_{U\times(0,r)} |\nabla V|^2 \phi t\, d\sigma dt + C(K)\varepsilon \int_U N_r^2(\nabla u) dy,$$

where $K > 0$ can be arbitrary small and $\varepsilon > 0$ depends on the Carleson norm of the coefficients. Thus as before by choosing K sufficiently small this term can be absorbed into the left-hand side of (3.12).

The third term of (3.22) is another "error" term of type similar to (3.13). We will handle this at the end. Hence the only term remaining is

$$-\sum_{k<n} \iint_{U\times(0,r)} \frac{b^k\partial_k a_{nj}}{a_{nn}} V v_j\, \phi\, d\sigma dt = -\sum_{k<n} \iint_{U\times(0,r)} \frac{b^k\partial_k a_{nj}}{a_{nn}} V v_j\, \phi \partial_n(t)\, d\sigma dt.$$

Here we introduced an extra term $1 = \partial_n(t)$ and now integrate by parts again. This gives

$$
\begin{aligned}
(3.23) \qquad & - \sum_{k<n} \int_{U\times\{r\}} \frac{b^k\partial_k a_{nj}}{a_{nn}} V v_j\, \phi t\, d\sigma dt + \\
& + \sum_{k<n} \iint_{U\times(0,r)} \partial_n\left(\frac{b^k}{a_{nn}}\right)(\partial_k a_{nj})V v_j\, \phi t\, d\sigma dt + \\
& + \sum_{k<n} \iint_{U\times(0,r)} \frac{b^k\partial_k a_{nj}}{a_{nn}} V \partial_n v_j\, \phi t\, d\sigma dt + \\
& + \sum_{k<n} \iint_{U\times(0,r)} \frac{b^k\partial_k a_{nj}}{a_{nn}} \partial_n V v_j\, \phi t\, d\sigma dt + \\
& + \sum_{k<n} \iint_{U\times(0,r)} \frac{b^k(\partial_n\partial_k a_{nj})}{a_{nn}} V v_j\, \phi t\, d\sigma dt.
\end{aligned}
$$

The first four terms are of same type we have encountered before. The first term here is cancelled by the last term of (3.21). The second term is bounded by $\varepsilon \int_U N_r^2(\nabla u)\, d\sigma$ (c.f. (3.10)). The third term is like (3.17) and the fourth like the second term of (3.22). Finally, in the last term we have two derivatives on the coefficient $(\partial_n\partial_k a_{nj})$ but only one of the derivatives is in the normal direction since $k < n$. Hence we integrate by parts one more time (moving the ∂_k derivative). We get

$$\begin{aligned}
&\sum_{k<n}\iint_{U\times(0,r)} \frac{b^k(\partial_n\partial_k a_{nj})}{a_{nn}} V v_j\, \phi t\, d\sigma dt = \\
-&\sum_{k<n}\iint_{U\times(0,r)} \partial_k\left(\frac{b^k}{a_{nn}}\right)(\partial_n a_{nj}) V v_j\, \phi t\, d\sigma dt - \\
-&\sum_{k<n}\iint_{U\times(0,r)} \frac{b^k\partial_n a_{nj}}{a_{nn}} V\, \partial_k v_j\, \phi t\, d\sigma dt - \\
-&\sum_{k<n}\iint_{U\times(0,r)} \frac{b^k\partial_n a_{nj}}{a_{nn}} \partial_k V v_j\, \phi t\, d\sigma dt - \\
-&\sum_{k<n}\iint_{U\times(0,r)} \frac{b^k\partial_n a_{nj}}{a_{nn}} V v_j\, (\partial_k\phi) t\, d\sigma dt.
\end{aligned} \tag{3.24}$$

Here the second, third and fourth terms are like the second, third and fourth terms in (3.23) and are handled likewise. Finally, the last term is another of the "error terms". This concludes the analysis of the term (3.16) in (3.12).

Finally, we sum over all choices of functions $V = V_\tau = \nabla u \cdot \vec{T}_\tau$, for $1 \le \tau \le m$ and over all sets U_ℓ (from Definition 3.1) choosing the smooth cutoff functions $\phi = \phi_\ell$ in (3.3) such that they are the partition of unity, that is

$$\sum \phi_\ell = 1 \text{ on } \partial\Omega \qquad \text{and} \qquad \operatorname{supp}\, \phi_\ell \subset \widetilde{U}_\ell.$$

We first observe that the terms we called "error terms" completely cancel out. There are the terms in (3.13) plus two extra terms later on. This is due to the fact that $\sum_\ell(\partial_j\phi_\ell) = 0$. That means that summing over τ these terms equal to zero. This cancellation happens even if we work on different coordinate charts since the term we started our calculation (3.3) does not depend on choice of coordinates. Hence after taking into account all remaining terms we have by (3.12):

$$\begin{aligned}
&\iint_{\partial\Omega\times(0,r)} |\nabla(\nabla_T u(X))|^2\delta(X)\, dX \approx \\
\approx\ &\sum_{i=1}^m \iint_{\partial\Omega\times(0,r)} |\nabla V_\tau(X)|^2\delta(X)\, dX \le \\
\le\ & K\int_{\partial\Omega} |\nabla_T u|^2\, d\sigma + \varepsilon\int_{\partial\Omega} N_r^2(\nabla u) d\sigma + \\
+\ & K\sum_{\tau=1}^m \left[\int_{\partial\Omega\times\{r\}} \partial_n(|V_\tau|^2) r\, d\sigma - \int_{\partial\Omega\times\{r\}} |V_\tau|^2\, d\sigma\right].
\end{aligned} \tag{3.25}$$

At this point we have to deal with the last two terms

$$\int_{\partial\Omega\times\{r\}} \partial_n(|V_\tau|^2) r\, d\sigma - \int_{\partial\Omega\times\{r\}} |V_\tau|^2\, d\sigma =$$
$$= \int_{\partial\Omega\times\{r\}} \partial_n(|V_\tau|^2 t)\, d\sigma - 2\int_{\partial\Omega\times\{r\}} |V_\tau|^2\, d\sigma \leq \int_{\partial\Omega\times\{r\}} \partial_n(|V_\tau|^2 t)\, d\sigma.$$

We would like to estimate this by a solid integral by integrating r over an interval $(0, r')$ and averaging. This yields

$$\frac{1}{r'}\int_0^{r'}\int_{\partial\Omega\times\{r\}} \partial_n(|V_\tau|^2 t)\, dX = \int_{\partial\Omega\times\{r'\}} |V_\tau|^2\, d\sigma. \tag{3.26}$$

This term is still not a solid integral so we use the averaging technique one more time by integrating over r' and averaging over an interval $(0, r_0)$. This yields a solid integral

$$\frac{1}{r_0}\iint_{\partial\Omega\times(0,r_0)} |V_\tau|^2\, dX \leq \frac{1}{r_0}\iint_{\partial\Omega\times(0,r_0)} |\nabla u|^2\, dX.$$

Going back to (3.25) we have to perform this double averaging procedure on all terms. This leads to introduction of some harmless weight terms and finally an estimate

$$\iint_{\partial\Omega\times(0,r/2)} |\nabla(\nabla_T u(X))|^2 \delta(X)\, dX \leq \tag{3.27}$$
$$\leq K\int_{\partial\Omega} |\nabla_T u|^2\, d\sigma + \varepsilon\int_{\partial\Omega} N_r^2(\nabla u) d\sigma + \frac{K}{r}\|\nabla u\|^2_{L^2(\Omega)}.$$

□

Lemma 3.2 deals with square function estimates for tangential directions. We have following for the normal derivative:

Lemma 3.3. *Let u be a solution to $Lu = \mathrm{div} A\nabla u = 0$, where L is an elliptic differential operator defined on Ω_{t_0} with bounded coefficients which are such that (3.1) is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then*

$$\int_{\partial\Omega} S_r^2(\partial_n u)\, d\sigma = \iint_{\partial\Omega\times(0,r)} |\nabla(\partial_n u(X))|^2 \delta(X)\, dX \leq \tag{3.28}$$
$$\leq K\iint_{\partial\Omega\times(0,r)} |\nabla(\nabla_T u(X))|^2 \delta(X)\, dX + \varepsilon\int_{\partial\Omega} N_r^2(\nabla u)\, d\sigma$$
$$= K\int_{\partial\Omega} S_r^2(\nabla_T u)\, d\sigma + \varepsilon\int_{\partial\Omega} N_r^2(\nabla u)\, d\sigma$$

provided $r \leq \min\{r_0, t_0\}$. Here $\varepsilon > 0$ depends only on the Carleson norm $C(r_0)$ and $\varepsilon \to 0+$ as $C(r_0) \to 0$ and K only depends on the domain, ellipticity constant and dimension n.

Proof. We integrate by parts in $\partial\Omega \times (0, r)$. We use the notation introduced above where we denoted $v_n = \partial_n u$. Clearly

$$\begin{aligned}
(3.29) \qquad & \iint_{\partial\Omega\times(0,r)} |\nabla v_n(X)|^2 \delta(X)\,dX \\
&= \iint_{\partial\Omega\times(0,r)} |\nabla_T v_n(X)|^2 \delta(X)\,dX + \iint_{\partial\Omega\times(0,r)} |\partial_n v_n(X)|^2 \delta(X)\,dX = \\
&= \iint_{\partial\Omega\times(0,r)} |\partial_n(\nabla_T u(X))|^2 \delta(X)\,dX + \iint_{\partial\Omega\times(0,r)} |\partial_n v_n(X)|^2 \delta(X)\,dX
\end{aligned}$$

The first term is clearly controlled by the square function of $\nabla_T u$. It remains to deal with the second term. Since

$$|a_{nn}\partial_n v_n|^2 = |\partial_n(a_{nn}v_n) - \partial_n(a_{nn})v_n|^2 \leq 2|\partial_n(a_{nn}v_n)|^2 + 2|\partial_n(a_{nn})v_n|^2.$$

We see that by the ellipticity assumption

$$\begin{aligned}
(3.30) \quad & \iint_{\partial\Omega\times(0,r)} |\partial_n v_n(X)|^2 \delta(X)\,dX \approx \iint_{\partial\Omega\times(0,r)} (a_{nn}(X))^2 |\partial_n v_n(X)|^2 \delta(X)\,dX \leq \\
&\leq 2\iint_{\partial\Omega\times(0,r)} |\partial_n(a_{nn}v_n)|^2 t\,dX + 2\iint_{\partial\Omega\times(0,r)} |\partial_n(a_{nn})v_n|^2 t\,dX.
\end{aligned}$$

Here as before $X = (x, t)$, i.e. t is the last n-th coordinate. The second term (using the Carleson condition) is bounded by $\varepsilon \int_{\partial\Omega} N_r^2(\nabla u)\,d\sigma$. We further estimate the first term. Using the equation u satisfies we see that

$$\partial_n(a_{nn}v_n) = -\sum_{(i,j)\neq(n,n)} \partial_i(a_{ij}\partial_j u).$$

From this point on we use local coordinates. It follows that

$$\begin{aligned}
(3.31) \quad & \iint_{\partial\Omega\times(0,r)} |\partial_n(a_{nn}v_n)|^2 t\,dX \leq (n^2-1)\sum_{(i,j)\neq(n,n)} \iint_{\partial\Omega\times(0,r)} |\partial_i(a_{ij}\partial_j u)|^2 t\,dX \\
&\leq 2(n^2-1)\sum_{(i,j)\neq(n,n)} \left[\iint_{\partial\Omega\times(0,r)} |\partial_i(a_{ij})|^2|\partial_j u|^2 t\,dX + \iint_{\partial\Omega\times(0,r)} |a_{ij}|^2|\partial_i\partial_j u|^2 t\,dX\right].
\end{aligned}$$

The first term here is of the same type as the last term of (3.30) and is bounded by $\varepsilon \int_{\partial\Omega} N_r^2(\nabla u)\,d\sigma$. Because $(i, j) \neq (n, n)$

$$|\partial_i\partial_j u|^2 \leq |\nabla(\nabla_T u)|^2,$$

hence the last term of (3.31) is also bounded by the square function of $\nabla_T u$. □

4. Comparability of the Nontangential Maximal Function and the Square Function

If we combine the results of Lemma 3.2 and 3.3 we obtain the following inequality.

Lemma 4.1. *Let u be a solution to $Lu = \operatorname{div} A\nabla u = 0$, where L is an elliptic differential operator defined on Ω_{t_0} with bounded coefficients which are such that (3.1) is the density of a Carleson measure on all Carleson boxes of size at most r_0.*

Then there exists $r_1 > 0$ and $K > 0$ depending only on the geometry of the domain Ω, ellipticity constant Λ, dimension n and the Carleson norm of coefficients such that

$$\int_{\partial\Omega} S^2_{r/2}(\nabla u)\, d\sigma \leq K \int_{\partial\Omega} N^2_r(\nabla u) d\sigma, \tag{4.1}$$

for all $r \leq \min\{r_0, r_1, t_0\}$.

Proof. We observe that first two terms on the righthand side of (3.2) can both be bounded by $K \int_{\partial\Omega} N^2_r(\nabla u) d\sigma$. Recall that the last term $\frac{K}{r}\|\nabla u\|^2_{L^2(\Omega)}$ appears there due to averaging of (3.26). This last averaging is however unnecessary as the righthand side of (3.26) can be directly bounded by a multiple of $\int_{\partial\Omega} N^2_r(\nabla u) d\sigma$. From this (4.1) follows. □

We note that Lemma 4.1 holds regardless of the size of Carleson norm $C(r_0)$. We would like to establish an analogue of Lemma 4.1 for values p different from 2. In order to do that we first observe that a local version of Lemma 4.1 is also true:

Lemma 4.2. *Consider an operator L defined on a subset $2U \times (0, r)$ of $\mathbb{R}^n_+$, with $r \simeq diam(U)$. Then there exists $K > 0$ depending only on the ellipticity constant Λ, dimension n and the Carleson norm of coefficients such that*

$$\int_{U\times(0,r)} |\nabla^2 u| t\, d\sigma\, dt \leq K \int_{2U} N^2_r(\nabla u) d\sigma. \tag{4.2}$$

Proof. The proof is essentially same as the proof of Lemma 4.1 since the estimate (3.2) is based on local considerations. However, the terms of type (3.13) have to be considered now as they only disappear in the global estimate. Observe that $|\partial_i \phi| \leq C/r$ hence these "error" terms are bounded from above by

$$C \iint_{U\times(0,r)} |\nabla^2 u||\nabla u| \tfrac{t}{r} d\sigma\, dt.$$

By Cauchy-Schwartz this can be further bounded by

$$C \left(\iint_{U\times(0,r)} |\nabla^2 u|^2 t\, d\sigma\, dt\right)^{1/2} \left(\iint_{U\times(0,r)} |\nabla u|^2 \tfrac{t}{r^2} d\sigma\, dt\right)^{1/2}.$$

Since $|\nabla u(X)| \leq N(\nabla u)(Q)$ for all $X \in \Gamma(Q)$ the term $\iint_{U\times(0,r)} |\nabla u|^2 \frac{t}{r^2} d\sigma\, dt$ is further bounded by

$$\int_U \frac{1}{r}\left(\int_0^r N^2_r(\nabla u)(Q) \tfrac{t}{r} dt\right) d\sigma(Q) \leq \int_U N^2_r(\nabla u) d\sigma.$$

From this (4.2) follows. □

We claim that Lemma 4.2 implies that the square function is controlled by the non-tangential maximal function in L^p for $p > 2$ as well.

Lemma 4.3. *Let u be a solution to $Lu = \mathrm{div} A\nabla u = 0$, where L is an elliptic differential operator defined on Ω_{t_0} with bounded coefficients which are such that (3.1) is the density of a Carleson measure on all Carleson boxes of size at most r_0.*

Then for any $p \geq 2$ there exists $r_1 > 0$ and $K > 0$ depending only on the geometry of the domain Ω, ellipticity constant Λ, dimension n and the Carleson norm of coefficients such that

$$\int_{\partial\Omega} S^p_{r/2}(\nabla u)\, d\sigma \leq K \int_{\partial\Omega} N^p_r(\nabla u) d\sigma, \tag{4.3}$$

for all $r \leq \min\{r_0, r_1, t_0\}$.

Proof. The lemma has already been proved when $p = 2$, since then it is just the statement of Lemma 4.1, so we only need to consider $p > 2$. Moreover, it suffices to prove (4.3) on each coordinate patch U_ℓ for $\ell = 1, 2, \dots, k$. In fact, we can go slightly further and say it is sufficient to prove

$$\int_{U^0_\ell} S^p_{r/2}(\nabla u)\, d\sigma \leq K \int_{U^0_\ell} N^p_r(\nabla u) d\sigma \tag{4.4}$$

for each ℓ, where $\widetilde{U}_\ell \subseteq U^0_\ell \subseteq U_\ell$. Because we only need to consider $p > 2$, Lemma 2 on page 152 of [18] shows that to prove (4.4) it is sufficient to show the relative distributional inequality

$$\begin{aligned} &|\{x \in U^0_\ell \,|\, S_{[a],r/2}(\nabla u)(x) > 2\lambda, M(N_r(\nabla u)^2)(x)^{\frac{1}{2}} \leq \alpha\lambda\}| \\ &\leq C\alpha^2 |\{x \in U^0_\ell \,|\, S_{[2a],r/2}(\nabla u)(x) > \lambda\}|, \end{aligned} \tag{4.5}$$

where M is the Hardy-Littlewood maximal function on U^0_ℓ.

Now, for each ℓ, we describe a localised Whitney decomposition of the set where $S_{[2a],r/2}(\nabla u) > \lambda$ (c.f. [9, A-34], which we follow here). First for each ℓ we find a finite number of cubes $Q_{\ell,j}$ for which $\widetilde{U}_\ell \subseteq \cup_j Q_{\ell,j} \subseteq U_\ell$ and the side length $\ell(Q_{\ell,j})$ of $Q_{\ell,j}$ is comparable with r. We denote $P_{\ell,j} = \{x \in Q_{\ell,j} \,|\, S_{[2a],r/2}(\nabla u)(x) > \lambda\}$ and $K_{\ell,j} = \{x \in Q_{\ell,j} \,|\, S_{[2a],r/2}(\nabla u)(x) \leq \lambda\}$

Fix a pair (ℓ, j). If $K_{\ell,j}$ is empty, define $\mathcal{F}_{\ell,j} := \{Q_{\ell,j}\}$. If $K_{\ell,j}$ is non-empty we will define $\mathcal{F}_{\ell,j}$ to be a collection of dyadic sub-cubes of $Q_{\ell,j}$ in the following way. First observe that we can write $P_{\ell,j}$ as the union of

$$P^k_{\ell,j} = \{x \in P_{\ell,j} \,|\, 2\ell(Q_{\ell,j})\sqrt{n}2^{-k} < \operatorname{dist}(x, K_{\ell,j}) \leq 4\ell(Q_{\ell,j})\sqrt{n}2^{-k}\}$$

for $k \in \mathbb{N}$.

We can find 2^{n-1} dyadic sub-cubes of $Q_{\ell,j}$ by bisecting each side of $Q_{\ell,j}$. We denote the collection of these 2^{n-1} cubes as $D^1_{\ell,j}$ and each cube in the collection has side length equal to $\ell(Q_{\ell,j})/2$. Equally, we can find 2^{n-1} dyadic sub-cubes of each cube in $D^1_{\ell,j}$ by again bisecting each side of it. Thus, we have $2^{2(n-1)}$ subcubes of the cubes in $D^1_{\ell,j}$ which have $\ell(Q_{\ell,j})/2^2$. We denote the collection of these $2^{2(n-1)}$ cubes by $D^2_{\ell,j}$. Continuing inductively $D^k_{\ell,j}$ is a collection of $2^{k(n-1)}$ dyadic cubes with side length equal to $\ell(Q_{\ell,j})/2^k$.

Let $\mathcal{F}'_{\ell,j}$ be the collection of all cubes Q in $D^k_{\ell,j}$ for some $k \in \mathbb{N}$ such that $Q \cap P^k_{\ell,j} \neq \emptyset$. Let $Q \in \mathcal{F}'_{\ell,j}$ and pick $x \in Q \cap P^k_{\ell,j}$. Observe that

$$\begin{aligned} \ell(Q_{\ell,j})2^{-k}\sqrt{n-1} &= \operatorname{dist}(x, K_{\ell,j}) - \ell(Q_{\ell,j})2^{-k}\sqrt{n-1} = \operatorname{dist}(x, K_{\ell,j}) - \ell(Q)\sqrt{n-1} \\ &\leq \operatorname{dist}(Q, K_{\ell,j}) \leq \operatorname{dist}(x, K_{\ell,j}) \leq 4\ell(Q_{\ell,j})2^{-k}\sqrt{n-1} \end{aligned}$$

and so,

$$\ell(Q)\sqrt{n-1} \leq \operatorname{dist}(Q, K_{\ell,j}) \leq 4\ell(Q)\sqrt{n-1}. \tag{4.6}$$

Given that $\mathcal{F}'_{\ell,j}$ is a collection of dyadic cubes, any two cubes which intersect have the property that one is contained in the other. Thus, we may define $\mathcal{F}_{\ell,j}$ to be the set of cubes $Q \in \mathcal{F}'_{\ell,j}$ such that if $Q' \in \mathcal{F}'_{\ell,j}$ and $Q \cap Q' \neq \emptyset$, then $Q' \subseteq Q$. That is $\mathcal{F}_{\ell,j}$ is the set of maximal cubes in $\mathcal{F}'_{\ell,j}$. Clearly then, $\mathcal{F}_{\ell,j}$ is a collection of disjoint dyadic cubes.

Our Whitney decomposition of $\{x \in U_\ell \,|\, S_{[a],r/2}(\nabla u)(x) > \lambda\}$ is then the collection

$$\mathcal{F}_\ell := \bigcup_j \mathcal{F}_{\ell,j}.$$

This collection has the properties that each $Q \in \mathcal{F}_\ell$ is such that either (4.6) holds or $\ell(Q) \simeq r$,

$$\bigcup_{Q \in \mathcal{F}_\ell} Q = \{x \in \cup_j Q_{\ell,j} \,|\, S_{[2a],r/2}(\nabla u)(x) > \lambda\},$$

and there exists a constant C such that there are at most C cubes that intersect at any given point.

Fix $Q \in \mathcal{F}_\ell$ and set

$$R := \{x \in Q \,|\, S_{r/2}(\nabla u)(x) > 2\lambda, M(N_r(\nabla u)^2)(x)^{\frac{1}{2}} \leq \alpha\lambda\}$$

If $x \in R$ and (4.6) holds for Q, then there exists x' such that $\operatorname{dist}(x, x') \leq 4\sqrt{n}\ell(Q)$ and $S_{[2a],r/2}(\nabla u)(x') \leq \lambda$. Consequently there exists a constant α such that

$$\begin{aligned} S_{[a],\alpha\ell(Q)}(\nabla u)^2(x) &\geq S_{[a],r/2}(\nabla u)^2(x) - \iint_{\Gamma_{[a],r/2}(x)\cap(\mathbb{R}^{n-1}\times(\alpha\ell(Q),r/2))} |\nabla^2 u|^2 t^{2-n} d\sigma dt \\ &\geq S_{[a],r/2}(\nabla u)^2(x) - S_{[2a],r/2}(\nabla u)^2(x') \\ &\geq 4\lambda^2 - \lambda^2 = 3\lambda^2 \end{aligned} \tag{4.7}$$

Then, if R is non-empty (say $x_0 \in R$), we can apply Lemma 4.2 to conclude that

$$\begin{aligned} |R| &\leq \frac{1}{3\lambda^2}\int_Q S_{[a],\alpha\ell(Q)}(\nabla u)^2 d\sigma \leq \frac{C}{\lambda^2}\int_{Q\times(0,\alpha\ell(Q))} |\nabla^2 u|^2 t \, d\sigma dt \\ &\leq \frac{CK}{\lambda^2}\int_{2Q} N_{\alpha\ell(Q)}(\nabla u)^2 d\sigma \leq \frac{2^n CK|Q|}{\lambda^2} M(N_r(\nabla u)^2)(x_0) \leq 2^n CK\alpha^2|Q|. \end{aligned} \tag{4.8}$$

Furthermore, if (4.6) does not hold, then $r/2 \simeq \ell(Q)$, so we may repeat (4.8) without the need for (4.7). Finally, we observe that the inequality $|R| \leq 2^n CK\alpha^2|Q|$ is trivial if R is empty. Thus, summing over $Q \in \mathcal{F}_\ell$ we obtain (4.5) with $U^0_\ell = \cup_j Q_{\ell,j}$. □

Now we would like to establish a reverse inequality showing that the non-tangential maximal function can be dominated by the square function. As we shall see in the proof we will have to assume *small* Carleson norm. We start with the following local lemma working in coordinates on $\mathbb{R}^n_+$ with boundary $\mathbb{R}^{n-1}$.

Lemma 4.4. *Let u be a solution to $Lu = \operatorname{div} A\nabla u = 0$, where L is an elliptic differential operator defined on $\mathbb{R}^n_+$ with bounded coefficients such that (3.1) is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm ε.*

Let ϕ be a non-negative Lipschitz function and let Q be a cube in $\mathbb{R}^{n-1}$ with $r = diam(Q)$. Suppose that $\phi(x) \le 12r/a$ for $x \in Q^*$. Here Q^* is a dilated Q by factor of 5. Then if $\|\nabla\phi\|_{L^\infty(\mathbb{R}^{n-1})}$ is sufficiently small, there are constants a (c.f. Definition 2.5) and C (depending only on Λ, Carleson norm ε, $\|\nabla\phi\|_{L^\infty(\mathbb{R}^{n-1})}$ and a), such that

$$\begin{aligned} \|\nabla u(.,\phi(.))\|^2_{L^2(Q)} \le \;& C \quad (\|S(\nabla u)\|^2_{L^2(Q^*)} + \varepsilon\|N(\nabla u)\|^2_{L^2(Q^*)} \\ &+ \quad \|N(\nabla u)\|_{L^2(Q^*)}\|S(\nabla u)\|_{L^2(Q^*)} + r^{n-1}|\nabla u(X_r)|^2), \end{aligned} \tag{4.9}$$

where X_r is an arbitrary corkscrew point, i.e., any point in $\{X = (x,t); \phi(x)+r/2 \le t \le \phi(x) + 6r/a\}$. The square and non-tangential maximal function in (4.9) are defined using non-tangential cones $\Gamma_a(.)$. Both square function and non-tangential maximal functions on the righthand side can be truncated at a height that is a multiple of r.

Proof. Recall the mapping $\Phi : \mathbb{R}^n_+ \to \Omega_\phi = \{X = (x,t); t > \phi(x)\}$ used by Dahlberg, Keing and Stein (see for example [2] or [15] and many others) defined as

$$\Phi(X) = (x, c_0 t + (\theta_t * \phi)(x)), \tag{4.10}$$

where $(\theta_t)_{t>0}$ is smooth compactly supported approximate identity and c_0 can be chosen large enough (depending only on $\|\nabla\phi\|_{L^\infty(\mathbb{R}^{n-1})}$ so that Φ is one to one. We pull back the solution u in Ω_ϕ of $\mathrm{div}(A\nabla u) = 0$ to a solution $v = u \circ \Phi$ of a different second order elliptic equation $\mathrm{div}(B\nabla v) = 0$.

The coefficient matrix B satisfies ellipticity condition with constant that is a multiple of Λ and which depends on $\|\nabla\phi\|_{L^\infty(\mathbb{R}^{n-1})}$. Also if ε is the Carleson norm of

$$\sup\{t|\nabla a_{ij}(Y)|^2 \;:\; Y \in B_{t/2}((x,t))\},$$

then the Carleson norm of

$$\sup\{t|\nabla b_{ij}(Y)|^2 \;:\; Y \in B_{t/2}((x,t))\},$$

for $B = (b_{ij})$ will only depend on ε and $\|\nabla\phi\|_{L^\infty}$. Furthermore, if $\|\nabla\phi\|_{L^\infty}$ is small enough, then the Carleson norm of the matrix B can be guaranteed to be less than 2ε.

We choose a smooth function $\xi_1 : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\xi_1(x) = 1$ for $x \in Q$, $|\xi_1'| \le 16/r$ and support contained in a concentric dilation $(9/8)Q$. Choose another function $\xi_2 : [0,\infty) \to \mathbb{R}$ such that $\xi_2(t) = 1$ on $[0,r]$, $|\xi_2'| \le 5/r$ and support contained in $[0, 2r]$. Now define $\xi(X) = \xi(x,t) = \xi_1(x)\xi_2(t)$.

Denote by $w_i = \partial_i v$ for $i = 1, 2, \dots, n$. For each $i \le n-1$ we have

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} w_i(x,0)^2\xi_1(x)dx &= -\iint_{\mathbb{R}^n_+} \partial_n(w_i^2\xi)(X)\,dX \\ &= -\iint_{\mathbb{R}^n_+} 2w_i(\partial_n w_i)\xi\,dX - \iint_{\mathbb{R}^n_+} w_i^2\xi_1\xi_2'\,dX. \end{aligned} \tag{4.11}$$

The second term on the right-hand side of (4.11) is controlled by $r^{-1}\iint_K w_i^2$ where $K = \{X = (x,t); x \in Q^* \text{ and } r/3 \le t \le 7r/a\}$. Let X_r be any point in K and choose K' and K'' to be the appropriate concentric enlargements of K. We set $c = \frac{1}{K'}\iint_{K'} w_i$. Using [8, Thm 8.17] we may further estimate this term by

$$
\begin{aligned}
& r^{-1}\iint_K (w_i - w_i(X_r))^2\,dX + r^{-1}\iint_K w_i^2(X_r)\,dX \\
\leq\; & Cr^{n-1}\operatorname{osc}_K(w_i)^2 + Cr^{n-1}|w_i(X_r)|^2 \\
\leq\; & Cr^{n-1}\sup_K |w_i - c|^2 + Cr^{n-1}|w_i(X_r)|^2 \\
\leq\; & Cr^{-1}\iint_{K'} |w_i - c|^2\,dX + Cr^{n-1+2(1-n/q)}\|(\partial_i B)\mathbf{w}\|^2_{L^q(K')} + Cr^{n-1}|w_i(X_r)|^2,
\end{aligned}
$$

for $q > n$. Here we are using (3.14) with matrix A replaced by B. Using the Poincaré's inequality and Carleson condition for B this can be further estimated by

$$
\begin{aligned}
& C\iint_{K''} |\nabla(w_i)|^2 r\,dX + C\varepsilon^2\|N(\nabla u)\|^2_{L^2(Q^*)} + Cr^{n-1}|w_i(X_r)|^2 \\
\leq\; & C\|S(\nabla u)\|^2_{L^2(Q^*)} + C\varepsilon^2\|N(\nabla u)\|^2_{L^2(Q^*)} + Cr^{n-1}|w_i(X_r)|^2.
\end{aligned} \tag{4.12}
$$

The first term on the righthand side of (4.11) can be estimated by

$$
\begin{aligned}
-\; & \iint_{\mathbb{R}^n_+} 2w_i(\partial_n w_i)\xi\,dX \\
=\; & -\iint_{\mathbb{R}^n_+} 2w_i(\partial_n w_i)\xi\partial_n(t)\,dX = 2\iint_{\mathbb{R}^n_+} [\partial_n(w_i(\partial_n w_i)\xi)]t\,dX \\
=\; & 2\iint_{\mathbb{R}^n_+} (\partial_n w_i)^2\xi t\,dX + 2\iint_{\mathbb{R}^n_+} w_i(\partial_n^2 w_i)\xi t\,dX + 2\iint_{\mathbb{R}^n_+} w_i(\partial_n w_i)\xi_1\xi_2' t\,dX \\
& =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Using the fact that $i \leq n-1$ we see that $\partial_n^2 w_i$ in the term II can be written as $\partial_i\partial_n w_n$. This gives

$$
\begin{aligned}
\mathrm{II} &= -2\iint_{\mathbb{R}^n_+} (\partial_n w_n)\partial_i(w_i\xi)t\,dX \\
&= -2\iint_{\mathbb{R}^n_+} (\partial_n w_n)\partial_i(w_i)\xi t\,dX - 2\iint_{\mathbb{R}^n_+} (\partial_n w_n)w_i\partial_i(\xi_1)\xi_2 t\,dX \\
&= \mathrm{II}_1 + \mathrm{II}_2.
\end{aligned}
$$

We observe that the terms I and II_1 are both bounded by the square function $\|S_{2r}(\mathbf{w})\|^2_{L^2(Q^*)}$. This is further bounded by $\|S(\nabla u)\|^2_{L^2(Q^*)}$, where the square function is truncated at a greater height or not truncated at all. For II_2 and III we have

$$\begin{aligned}\mathrm{II}_2+\mathrm{III} &\le \frac{C}{r}\iint_{Q^*\times(0,2r)}|\nabla\mathbf{w}||\mathbf{w}|t\,dX\\ &\le C\left(\iint_{Q^*\times(0,2r)}|\nabla\mathbf{w}|^2t\,dX\right)^{1/2}\left(\iint_{Q^*\times(0,2r)}|\mathbf{w}|^2\tfrac{t}{r^2}\,dX\right)^{1/2}\\ &\le C\|S(\mathbf{w})\|_{L^2(Q^*)}\left(\int_{Q^*}\frac{1}{r}\int_0^{2r}|\mathbf{w}|^2\,dt\,dx\right)^{1/2}\\ &\le C\|S(\mathbf{w})\|_{L^2(Q^*)}\left(\int_{Q^*}\frac{2r}{r}|N(\mathbf{w})|^2\,dx\right)^{1/2}=C\|S(\mathbf{w})\|_{L^2(Q^*)}\|N(\mathbf{w})\|_{L^2(Q^*)}.\end{aligned}$$

This bounds (4.11) by terms that appear on the righthand side of (4.9).

It remains to estimate $\int_{\mathbb{R}^{n-1}}w_n(x,0)^2\xi_1(x)dx$. We estimate instead an expression for co-normal derivative $H=\sum_j b_{nj}w_j$. This is sufficient since

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}w_n(x,0)^2\xi_1(x)dx &\approx \int_{\mathbb{R}^{n-1}}(b_{nn}w_n)^2(x,0)\xi_1(x)dx\\ &\le n\left[\int_{\mathbb{R}^{n-1}}H^2\xi_1\,dx+\sum_{j<n}\int_{\mathbb{R}^{n-1}}(b_{nj}w_j)^2\xi_1\,dx\right]\\ &\le n\int_{\mathbb{R}^{n-1}}H^2\xi_1\,dx+C\sum_{j<n}\int_{\mathbb{R}^{n-1}}w_j^2(x,0)\xi_1(x)\,dx\end{aligned}\tag{4.13}$$

Hence if we can obtain estimates for the first term we are done since the second term has already been bounded. We proceed as before.

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}H(x,0)^2\xi_1(x)dx &= -\iint_{\mathbb{R}^n_+}\partial_n(H^2\xi)(X)\,dX\\ &= -\iint_{\mathbb{R}^n_+}2H(\partial_nH)\xi\,dX-\iint_{\mathbb{R}^n_+}H^2\xi_1\xi_2'\,dX.\end{aligned}\tag{4.14}$$

As before we observe that the second term can be bounded by $r^{-1}\sum_i\iint_K w_i^2$. The calculation we have done above holds for any i even $i=n$ giving us bound (4.12).

It remains to deal with the first term. Using the equation $\operatorname{div}(B\nabla v)=0$

$$\partial_nH=\sum_j\partial_n(b_{nj}\partial_jv)=-\sum_{i<n}\partial_i(b_{ij}\partial_jv)=-\sum_{i<n}\partial_i(b_{ij}w_j).$$

It follows that

$$
\begin{aligned}
& - \iint_{\mathbb{R}^n_+} 2H(\partial_n H)\xi \, dX \\
& = \sum_{i<n} \iint_{\mathbb{R}^n_+} 2H\partial_i(b_{ij}w_j)(\partial_n t)\xi \, dX = -\sum_{i<n} \iint_{\mathbb{R}^n_+} 2\partial_n(H\partial_i(b_{ij}w_j)\xi) t \, dX \qquad (4.15)\\
& = -\sum_{i<n} \iint_{\mathbb{R}^n_+} 2(\partial_n H)\partial_i(b_{ij}w_j)\xi t \, dX - \iint_{\mathbb{R}^n_+} 2H\partial_i\partial_n(b_{ij}w_j)\xi t \, dX \\
& \quad - \iint_{\mathbb{R}^n_+} 2H\partial_i(b_{ij}w_j)\xi_1\xi_2' t \, dX = \widetilde{\mathrm{I}} + \widetilde{\mathrm{II}} + \widetilde{\mathrm{III}}.
\end{aligned}
$$

As before we do further integration by parts for the term $\widetilde{\mathrm{II}}$.

$$
\begin{aligned}
\widetilde{\mathrm{II}} &= \iint_{\mathbb{R}^n_+} 2\partial_n(b_{ij}w_j)\partial_i(H\xi) t \, dX \\
&= 2\iint_{\mathbb{R}^n_+} 2\partial_n(b_{ij}w_j)(\partial_i H)\xi t \, dX + \iint_{\mathbb{R}^n_+} 2\partial_n(b_{ij}w_j)H(\partial_i\xi_1)\xi_2 t \, dX \\
&= \widetilde{\mathrm{II}_1} + \widetilde{\mathrm{II}_2}.
\end{aligned}
$$

We observe that when the derivative in terms $\widetilde{\mathrm{II}_2}$ and $\widetilde{\mathrm{III}}$ does not hit the coefficients b_{ij} these can be estimated exactly as the corresponding terms II_2 and III. When the derivative falls on the coefficient we get "error terms" that can be estimated using the Carleson measure property of the coefficients. In particular the term from $\widetilde{\mathrm{III}}$ is of the same form as (3.20) and is handled analogously. The term we obtain from $\widetilde{\mathrm{II}_2}$ is of a different nature and can be bounded above by

$$
\iint_{Q^*\times(0,2r)} |\mathbf{w}|^2 |\nabla B| \tfrac{t}{r} \, dX.
$$

By Cauchy-Schwarz this is no more than

$$
\begin{aligned}
& C\left(\iint_{Q^*\times(0,2r)} |\nabla B|^2 |\mathbf{w}|^2 t \, dX\right)^{1/2} \left(\iint_{Q^*\times(0,2r)} |\mathbf{w}|^2 \tfrac{t}{r^2} \, dX\right)^{1/2} \\
\leq\ & C\varepsilon \|N(\mathbf{w})\|_{L^2(Q^*)} \left(\int_{Q^*} \frac{1}{r}\int_0^{2r} |\mathbf{w}|^2 \, dt \, dx\right)^{1/2} \\
\leq\ & C\varepsilon \|N(\mathbf{w})\|_{L^2(Q^*)} \left(\int_{Q^*} \frac{2r}{r} |N(\mathbf{w})|^2 \, dx\right)^{1/2} = C\varepsilon \|N(\mathbf{w})\|^2_{L^2(Q^*)}.
\end{aligned}
$$

The terms $\widetilde{\mathrm{I}}$ and $\widetilde{\mathrm{II}_1}$ contain both a derivative acting on H and a derivative acting on $b_{ij}w_j$. We deal with these in two parts: (a) when the derivative acting on $H = \sum b_{nj}w_j$ falls on b_{nj} and (b) when it falls on w_j. First we deal with case (b). When the derivative acting on $b_{ij}w_j$ does not hit the coefficients, we can handle them as the corresponding terms I and II_1. When this derivative falls on the coefficients, the term we get from $\widetilde{\mathrm{I}}$ is again of the same nature as (3.20) and the term we get from $\widetilde{\mathrm{II}_1}$ looks like (3.9), so these terms are handled as before. Finally we deal with case (a), where we either get terms of the form (3.9), which we have dealt with before, or terms of the form

$$\iint_{Q^*\times(0,2r)} |\mathbf{w}|^2|\nabla B|^2 t\xi dX \lesssim \varepsilon \int_{Q^*} N_r^2(\nabla u)d\sigma. \tag{4.16}$$

This concludes the proof as

$$\|\nabla u(.,\phi(.))\|^2_{L^2(Q)} \leq C\sum_{i=1}^n \int_{\mathbb{R}^{n-1}} w_i(x,0)^2\xi_1(x)dx.$$

□

From now on we follow the stopping time argument from [14], in particular our Lemma 4.4 is an analogue of [14, Lemma 3.8]. For any continuous function $\mathbf{v}:\mathbb{R}^n_+\to\mathbb{R}^n$ and $\nu\in\mathbb{R}$ we define

$$h_{\nu,a}(\mathbf{v})(x) = \sup\{t\geq 0;\ \sup_{\Gamma_a(x,t)} |\mathbf{v}| > \nu\}.$$

Here $\Gamma_a(x,t)$ is a cone with vertex at (x,t) (recall that the boundary point is $(x,0)$). Hence

$$\Gamma_a(x,t) = (0,t) + \Gamma_a(x,0),$$

is the non-tangential cone $\Gamma_a(x,0)$ shifted in the direction $(0,t)$.

Lemma 4.5. *If $\mathbf{v}$ is such that $h_{\nu,a}(\mathbf{v})<\infty$ then $h_{\nu,a}(\mathbf{v})$ is Lipschitz with constant $1/a$.*

Proof. See, for example [14, Lemma 3.13]. □

We also have an analogue of [14, 3.14].

Lemma 4.6. *Let u be a solution to $Lu = \mathrm{div}A\nabla u = 0$, where L is an elliptic differential operator defined on $\mathbb{R}^n_+$ with bounded coefficients which are such that (3.1) is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm ε. Set $\mathbf{v}=\nabla u$ and let $(Q_j)_j$ be a Whitney decomposition of $\{x;\ N_{[a]}(\mathbf{v})(x)>\nu/24\}$. Given $a>0$, let $E^j_{\nu,\rho}$ be the intersection of the cube Q_j with*

$$\{x;\ N_{[a/12]}(\mathbf{v})(x)>\nu \text{ and } \varepsilon N_{[a]}(\mathbf{v})(x)+S_{[a]}(\mathbf{v})(x)\leq\rho\nu\}.$$

Then there exist a sufficiently small choice of ρ, independent of Q_j so that for each $x\in E^j_{\nu,\rho}$ there is a cube R with $x\in 6R$ and $R\subset Q_j^*$ for which

$$|\mathbf{v}(z,h_{\nu,a/12}(\mathbf{v})(z))|>\nu/2$$

for all $z\in R$.

Proof. Let $x\in E^j_{\nu,\rho}$. By definition $h_{\nu,a/12}(\mathbf{v})(x)>0$ and so there exists a Y on $\partial\Gamma_{a/12}(x,h_{\nu,a/12}(\mathbf{v})(x))$ such that $|\mathbf{v}(Y)|=\nu$ (here $Y=(y,y_n)$) and $h_{\nu,a/12}(\mathbf{v})(y)=y_n$. Let $r_0=y_n$ and

$$K=\Gamma_{a/12}(x,0)\cap\{Z;\ |z_n-y_n|<r_0/6\}.$$

Since Q_j is a Whitney cube, $r_0\leq(1+4\sqrt{n-1})\ell(Q_j)/a$, and we also have

$$3K\subset\Gamma_a(x,0)\quad\text{and}\quad \mathrm{dist}(3K,\partial\mathbb{R}^n_+)\geq r_0/2.$$

Hence again by [8, Thm 8.17] we have that

$$\mathrm{osc}_K(\mathbf{v}) \leq C(r_0^{-n/2}\|\mathbf{v}-\mathbf{c}\|_{L^2(2K)} + r_0^{1-n/q}\|(\nabla A)\mathbf{v}\|_{L^q(2K)}),$$

for any constant $\mathbf{c}$ and $q > n$. By (3.1) $|(\nabla A)\mathbf{v}|(Z) \leq Cr_0^{-1}\varepsilon N_{[a]}(\mathbf{v})(x)$ for $Z \in 2K$, so

$$r_0^{1-n/q}\|(\nabla A)\mathbf{v}\|_{L^q(2K)} \leq C\varepsilon N_{[a]}(\mathbf{v})(x)$$

and so using Poincaré's inequality

$$\begin{aligned} |\mathbf{v}(Z)-\mathbf{v}(Y)| &\leq \mathrm{osc}_K(\mathbf{v}) \leq C(r_0^{1-n/2}\|\nabla\mathbf{v}\|_{L^2(3K)} + \varepsilon N_{[a]}(\mathbf{v})(x)) \\ &\leq C(S_{[a]}(\mathbf{v})(x) + \varepsilon N_{[a]}(\mathbf{v})(x)) \leq C\rho\nu, \end{aligned}$$

for any $Z \in K$. Thus we may choose ρ sufficiently small so that $|\mathbf{v}(Z)-\mathbf{v}(Y)| \leq \nu/2$. Then clearly $|\mathbf{v}(z, h_{\nu,a/12}(\mathbf{v})(z))| > \nu/2$ for $|z-y| \leq ar_0/72$. □

Finally, the good-λ inequality can be converted using standard methods to a global inequality between N and S. We have the following:

Lemma 4.7. *Let u be a solution to $Lu = \mathrm{div}A\nabla u = 0$, where L is an elliptic differential operator defined on Ω_{t_0} with bounded coefficients such that*

$$\sup\{\delta(X)|\nabla a_{ij}(Y)|^2 \ :\ Y \in B_{\delta(X)/2}(X)\} \tag{4.17}$$

is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$.

Then there exists $\varepsilon > 0$ depending only on the geometry of the domain Ω, the ellipticity constant Λ, dimension n and p such that if $C(r_0) < \varepsilon$ then

$$\int_{\partial\Omega} N^2_{r/2}(\nabla u)\,dx \leq K\int_{\partial\Omega} S^2_r(\nabla u)dx + \iint_{\Omega\setminus\Omega_{r/2}} |\nabla u|^2\,dX. \tag{4.18}$$

Here K only depends on the geometry of the domain Ω, elliptic constant Λ, p and dimension n. Here N_h, S_h are truncated versions of non-tangential maximal function and square function, respectively.

Remark. The term $\iint_{\Omega\setminus\Omega_{r/2}} |\nabla u|^2\,dX$ is necessary if Ω is a bounded domain. Consider for example $L = \Delta$ on $\Omega \subset \mathbb{R}^n$. Let u be a harmonic function in Ω. Then for any vector $\mathbf{c}$ we have that $S(\nabla u) = S(\nabla(u + \mathbf{c}\cdot\mathbf{x}))$ but clearly $N(\nabla u) \neq N(\nabla(u + \mathbf{c}\cdot\mathbf{x}))$. This term is not necessary if the domain is unbounded and we consider untruncated versions of the non-tangential maximal function and the square function.

Proof. We only highlight the major points of the proof as the basic idea is the same as in [14]. Applying standard good-λ inequality methods and Lemma 4.6, we see that

$$\begin{aligned} &|\{x; N_{[a/12]}(\nabla u)(x) > \lambda\}| \\ \leq\ &|\{x; N_{[a/12]}(\nabla u)(x) > \lambda, \varepsilon N_{[a]}(\nabla u)(x) + S_{[a]}(\nabla u)(x) \leq \rho\lambda\}| \\ &+|\{x; \varepsilon N_{[a]}(\nabla u)(x) + S_{[a]}(\nabla u)(x) > \rho\lambda\}| \\ \leq\ &|\{x; M(\nabla u(\cdot, h_{\nu,a/12}(\nabla u)(\cdot))(x) > \lambda/2\}| \\ &+|\{x; \varepsilon N_{[a]}(\nabla u)(x) + S_{[a]}(\nabla u)(x) > \rho\lambda\}| \end{aligned}$$

Multiplying this inequality by λ and integrating in λ, we obtain the inequality

$$\begin{aligned}
&\|N_{[a/12]}(\mathbf{v})\|^2_{L^2(Q_0)}\\
\leq\ & C\left(\|M(\mathbf{v}(\cdot,h_{\nu,a/12}(\mathbf{v})(\cdot)))\|^2_{L^2(Q_0)}+\|S_{[a]}(\mathbf{v})\|^2_{L^2(Q_0)}+\varepsilon\|N_{[a]}(\mathbf{v})\|^2_{L^2(Q_0)}\right)\\
\leq\ & C\left(\|\mathbf{v}(\cdot,h_{\nu,a/12}(\mathbf{v})(\cdot))\|^2_{L^2(Q)}+\|S_{[a]}(\mathbf{v})\|^2_{L^2(Q^*)}+\varepsilon\|N_{[a]}(\mathbf{v})\|^2_{L^2(Q^*)}\right).
\end{aligned}\tag{4.19}$$

Here M is the Hardy-Littlewood maximal function, Q_0 is a cube, $Q=2Q_0$ and $Q^*=5Q$ is such that Q^* is contained in a single coordinate patch U_ℓ.

Now we can apply Lemma 4.4 to the first term on the right-hand side of (4.19) and after summing over an appropriate collection of cubes Q_0 obtain

$$\begin{aligned}
\|N_{[a/12]}(\nabla u)\|^2_{L^2(\partial\Omega)}\leq\ & C\quad(\|S_{[a]}(\nabla u)\|^2_{L^2(\partial\Omega)}+\varepsilon\|N_{[a]}(\nabla u)\|^2_{L^2(\partial\Omega)}\\
&+\quad\|N_{[a]}(\nabla u)\|_{L^2(\partial\Omega)}\|S_{[a]}(\nabla u)\|_{L^2(\partial\Omega)}+\iint_{\Omega\setminus\Omega_{r/2}}|\nabla u|^2\,dX),
\end{aligned}\tag{4.20}$$

Note that Lemma 4.4 requires the Lipschitz function ϕ has a small Lipschitz norm. Since we are using function $h_{\nu,a/12}(\mathbf{v})$ in place of ϕ, if we choose $a>0$ large enough by Lemma 4.5 the Lipschitz norm will be small. Standard techniques also tell us that $\|N_{[a]}(\nabla u)\|_{L^2(\partial\Omega)}\lesssim\|N_{[a/12]}(\nabla u)\|_{L^2(\partial\Omega)}$, so if ε is chosen small enough in (4.20) then (4.18) will follow. □

5. The $(R)_2$ Regularity Problem

Theorem 5.1. *Let $\Omega\subset M$ be a Lipschitz domain with Lipschitz norm L on a smooth Riemannian manifold M and $Lu=div(A\nabla u)$ be an elliptic differential operator defined on Ω with ellipticity constant Λ and coefficients which are such that*

$$\sup\{\delta(X)|\nabla a_{ij}(Y)|^2\ :\ Y\in B_{\delta(X)/2}(X)\}\tag{5.1}$$

is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon=\varepsilon(\Lambda,n)>0$ such that if $\max\{L,C(r_0)\}<\varepsilon$ then the regularity problem

$$\begin{aligned}
&Lu=0, &&\text{in }\Omega,\\
&u=f, &&\text{on }\partial\Omega,\\
&N(\nabla u)\in L^2(\partial\Omega),
\end{aligned}$$

is solvable for all f with $\|\nabla_T f\|_{L^2(\partial\Omega)}<\infty$. Moreover, there exists a constant $C=C(\Lambda,n,a)>0$ such that

$$\|N(\nabla u)\|_{L^2(\partial\Omega)}\leq C\|\nabla_T f\|_{L^2(\partial\Omega)}.\tag{5.2}$$

Proof. For any f in the Besov space $B^{2,2}_{1/2}(\partial\Omega)$ the exists a unique $H^2_1(\Omega)$ solution by the Lax-Milgram theorem. Observe that $f\in H^2_1(\partial\Omega)\subset B^{2,2}_{1/2}(\partial\Omega)$ so it only remains to establish the estimate (5.2).

Consider $\varepsilon>0$ and take $C(r_0)<\varepsilon$. To keep matters simple let us first consider the case when $\partial\Omega$ is smooth. In this case Lemma 3.2 applies directly. If follows that for all small r

$$\int_{\partial\Omega} S^2_{r/2}(\nabla u)\,d\sigma \le K\int_{\partial\Omega}|\nabla_T u|^2\,d\sigma + C(\varepsilon)\int_{\partial\Omega} N^2_r(\nabla u)d\sigma + \frac{K}{r}\|\nabla u\|^2_{L^2(\Omega)}. \tag{5.3}$$

Here $C(\varepsilon)\to 0$ if $\varepsilon\to 0$. We now choose ε small enough such that Lemma 4.7 holds. It follows that

$$\begin{aligned}\int_{\partial\Omega} N^2_{r/4}(\nabla u)\,d\sigma \le \widetilde{K}\int_{\partial\Omega}|\nabla_T u|^2\,d\sigma \quad &+ \quad C(\varepsilon)K^2\int_{\partial\Omega} N^2_r(\nabla u)d\sigma \\ &+ \quad \widetilde{K}(r)\|\nabla u\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.4}$$

Here K appearing in (5.4) is the same constant as in the estimate (4.18).

We also observe that we have a pointwise estimate

$$N^2_r(\nabla u)(X) \le N^2_{r/4}(\nabla u)(X) + C(r)\iint_{\Omega_{r/8}}|\nabla u(Y)|^2\,dY \tag{5.5}$$

for all $X\in\partial\Omega$. Seeing this is not hard as we are estimating $|\nabla u|$ away from the boundary. Hence by the Carleson condition we have $|\nabla A|\le C(r)$ there. Rest is a standard bootstrap argument using the equation $\mathbf{v}=\nabla u$ satisfies, i.e., $L\mathbf{v}=\operatorname{div}((\nabla A)\mathbf{v})$ eventually yielding pointwise bound on $|\nabla u|$ for $\{X\in\partial\Omega;\ \operatorname{dist}(X,\partial\Omega)\in[r/4,r]\}$. Finally, combining (5.4) and (5.5) we obtain

$$\begin{aligned}\int_{\partial\Omega} N^2_{r}(\nabla u)\,d\sigma \le \widetilde{K}\int_{\partial\Omega}|\nabla_T u|^2\,d\sigma \quad &+ \quad C(\varepsilon)K^2\int_{\partial\Omega} N^2_r(\nabla u)d\sigma \\ &+ \quad \widetilde{\widetilde{K}}(r)\|\nabla u\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.6}$$

We now can make our final choice of ε. We choose it sufficiently small such that the constant in (5.6) $C(\varepsilon)K^2<1/2$ which yields

$$\int_{\partial\Omega} N^2_{r}(\nabla u)\,d\sigma \le 2\widetilde{K}\int_{\partial\Omega}|\nabla_T u|^2\,d\sigma \quad + 2\widetilde{\widetilde{K}}(r)\|\nabla u\|^2_{L^2(\Omega)}. \tag{5.7}$$

From this the desired estimate follows since the term $\|\nabla u\|^2_{L^2(\Omega)}$, i.e., an $H^2_1(\Omega)$ estimate of the solution u, follows from the Lax-Milgram.

Now turn to the more general case, when Ω has a Lipschitz boundary with sufficiently small Lipschitz constant L. This case also includes the C^1 boundary as in such case L can be taken arbitrary small.

The crucial point is that the proofs of Lemmas 3.2-4.7 in the smooth case are based on local estimates near boundary $\partial\Omega$. This means we can reduce the matter to a situation where we have an open set U in $\mathbb{R}^n$ and a Lipschitz function ϕ with Lipschitz constant L such that in U the set Ω looks like $\{(x,t)\in\mathbb{R}^n; t>\phi(x)\}$.

Now the map (4.10) is a bijection between the sets $\mathbb{R}^n_+$ and $\{(x,t)\in\mathbb{R}^n; t>\phi(x)\}$. In fact if $c>\ell$ then the map Φ is a local bijection,, where $\ell=\|\nabla\phi\|_{BMO}$. Hence by pulling back everything (metric, coefficients) using Φ we are left with proving local estimates on a subset of $\mathbb{R}^n_+$. However, this is exactly what we did above. We only have to be careful about how much the Carleson constant of the coefficients changes when we move from the set $\{(x,t)\in\mathbb{R}^n; t>\phi(x)\}$ to $\mathbb{R}^n_+$. A computation gives us that if the original constant was C, the new constant on $\mathbb{R}^n_+$ will be $C+C(\ell)$ where $C(\ell)$ is an increasing function in ℓ such that $\lim_{\ell\to 0+}C(\ell)=0$. From this the claim follows, as this implies that $C + C(\ell)$ will be small as long as both C and ℓ are small enough. So we get solvability on domains with small Lipschitz constant, as well as on domains whose boundaries are given locally by functions with gradient in VMO. $\square$

Finally, we replace the gradient Carleson condition (5.1) by a weaker condition for oscillation of the coefficients (5.8). What this means is that the gradient ∇u no longer has well-defined pointwise non-tangential maximal function N. Instead a weaker version $\widetilde{N}$ defined by (2.1) must be used.

Theorem 5.2. *Under the same assumptions as in Theorem 5.1 the $(R)_2$ regularity problem for the operator $Lu = div(A\nabla u)$ is solvable under a weaker Carleson condition requiring that*

$$\delta(X)^{-1}\left(osc_{B(X,\delta(X)/2)}a_{ij}\right)^2 \tag{5.8}$$

is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0) < \varepsilon$.

Moreover, there exists a constant $C = C(\Lambda, n, a) > 0$ such that

$$\|\widetilde{N}(\nabla u)\|_{L^2(\partial\Omega)} \leq C\|\nabla_T f\|_{L^2(\partial\Omega)}. \tag{5.9}$$

Proof. The proof uses same idea as [5, Corollary 2.3]. For this reason we skip the non-essential details. The procedure outlined in [5] implies that for a matrix A satisfying (5.8) with ellipticity constant Λ one can find (by mollifying coefficients of A) a new matrix $\widetilde{A}$ with same ellipticity constant Λ such that $\widetilde{A}$ satisfies (5.1) and

$$\sup\{\delta(X)^{-1}|(A-\widetilde{A})(Y)|^2;\, Y \in B(X,\delta(X)/2)\} \tag{5.10}$$

is the density of a Carleson measure. Moreover, if the Carleson norm for matrix A is small (on balls of radius $\leq r_0$), so are the Carleson norms of (5.1) for $\widetilde{A}$ and (5.10). Hence by Theorem 5.1 the $(R)_2$ regularity problem is solvable for the operator $\widetilde{L}u = \mathrm{div}(\widetilde{A}\nabla u)$.

The solvability of the regularity problem for perturbed operators satisfying (5.10) has been studied in [13]. It follows by [13, Theorem 2.1] that the L^p regularity problem for the operator L is solvable for some $p > 1$. The p for which the solvability of the regularity problem is assessed is the p such that the $L^{p'}$, $p' = p/(p-1)$, Dirichlet problem for the adjoint operator L^* is solvable. Actually, the results in [13] are stated for symmetric operators, however careful study of the proof of [13, Theorem 2.1] reveals that what is really needed is to replace L by its adjoint when the $L^{p'}$ Dirichlet problem is considered.

However by [5, Theorem 2.2] the L^2 Dirichlet problem for L^* is solvable provided the Carleson norm of (5.8) (and hence (5.1) for $\widetilde{A}$) is sufficiently small. Hence we have solvability of the regularity problem $(R)_2$ by [13, Theorem 2.1, Remark 2.3]. $\square$

6. The Square Function Revisited

In this section we revisit bounds for the square function of ∇u from the perspective of the Neumann problem. As in Section 2 we shall assume that Ω is a smooth domain on a smooth compact Riemannian manifold M. We also continue to use notation we introduced there. Recall that Ω_{t_0} denotes the collar neighborhood of the boundary $\partial\Omega \times (0, t_0)$.

On Ω_{t_0} we have a well-defined co-normal derivative of u with respect to the operator L; in the metric $d\sigma \otimes dt$ this is just

$$H = \sum_{i=1}^{n} a_{ni}\partial_i u,$$

where (a_{ij}) are coefficients of the matrix A in local coordinates near the boundary.

We have the following key lemma bounding the non-tangential maximal function of ∇u by the square function of H.

Lemma 6.1. *Let $p \geq 2$ and u be a solution to $Lu = \mathrm{div} A\nabla u = 0$, where L is an elliptic differential operator defined on a smooth domain Ω with bounded coefficients such that (4.17) is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon > 0$ such that if $C(r_0) < \varepsilon$ then for some $K = K(\Omega, \Lambda, n, \varepsilon, p) > 0$*

$$\int_{\partial\Omega} N^p(\nabla u)\, dx \leq K \iint_{\Omega_{2r}} |\nabla_T u|^{p-2} |\nabla H|^2 \delta(X)\, dX + C(r) \iint_{\Omega \setminus \Omega_{r/2}} |\nabla u|^p\, dX. \tag{6.1}$$

Proof. We mainly work in the collar neighborhood Ω_{t_0} defined above. We choose $r \leq t_0/5$. Using the results we have on the solvability of the regularity problem we know that for sufficiently small $\varepsilon > 0$ we have:

$$\int_{\partial\Omega} N^p(\nabla u)\, dx \leq K \int_{\partial\Omega} |\nabla_T u|^p\, dx. \tag{6.2}$$

Since $\partial\Omega$ is a smooth compact manifold, there is a finite collection of balls $Q_1, Q_2, \dots, Q_k$ in $\mathbb{R}^{n-1}$ of diameter comparable to r and smooth diffeomorphisms $\varphi_\ell : 5Q_\ell \to \partial\Omega$ such that $\bigcup_\ell \varphi_\ell(9/8Q_\ell)$ covers $\partial\Omega$. Here rQ denotes the concentric enlargement of Q by a factor of r. Let us also find smooth partition of unity ϕ_ℓ such

$$\sum \phi_\ell = 1 \text{ on } \partial\Omega, \qquad \phi_\ell = 1 \text{ on } Q_\ell \qquad \text{and} \qquad \mathrm{supp}\ \phi_\ell \subset 9/8Q_\ell.$$

Let us fix ℓ and work on one ball $Q = Q_\ell$ and $\xi_1 = \phi_\ell$. We may assume that $|\xi_1'| \leq C/r$. Choose another function $\xi_2 : [0, \infty) \to \mathbb{R}$ such that $\xi_2(t) = 1$ on $[0, r]$, $|\xi_2'| \leq 5/r$ and support contained in $[0, 2r]$. Now define

$$\xi(X) = \xi(x, t) = \xi_1(x)\xi_2(t). \tag{6.3}$$

We work on estimating righthand side of (6.2) in local coordinates on $5Q \times (0, 5r)$. Denote by $v_k = \partial_k u$ for $k = 1, 2, \dots, n$. For each $k \leq n-1$ we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |v_k(x,0)|^p \xi_1(x) dx &= -\iint_{\mathbb{R}^n_+} \partial_n(|v_k|^p \xi)(X)\, dX \\ &= -p\iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k (\partial_n v_k)\xi\, dX - \iint_{\mathbb{R}^n_+} |v_k|^p \xi_1 \xi_2'\, dX = I + II. \end{aligned} \tag{6.4}$$

The second term on the right-hand side of (6.4) is controlled by $\iint_K |\nabla u|^p$ where $K = \{X = (x,t); x \in 5Q \text{ and } r/2 \leq t \leq 5r\}$. We deal with the first term. Since $\partial_n v_k = \partial_k v_n$ we have

$$
\begin{aligned}
I &= -p \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k (\partial_k v_n) \xi \, dX \\
&= -p \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k \partial_k \left(\frac{a_{ni}}{a_{nn}} v_i \right) \xi \, dX + p \sum_{i<n} \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k \partial_k \left(\frac{a_{ni}}{a_{nn}} v_i \right) \xi \, dX. \qquad (6.5)
\end{aligned}
$$

The second term of (6.5) can be further written as

$$
\begin{aligned}
& p \sum_{i<n} \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k \partial_k \left(\frac{a_{ni}}{a_{nn}} v_i \right) \xi \, dX \\
= \; & p \sum_{i<n} \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k v_i \partial_k \left(\frac{a_{ni}}{a_{nn}} \right) \xi \, dX + \sum_{i<n} \iint_{\mathbb{R}^n_+} \partial_i (|v_k|^p) \frac{a_{ni}}{a_{nn}} \xi \, dX. \qquad (6.6)
\end{aligned}
$$

We introduce $(\partial_n t)$ into both the terms of (6.6) and integrate by parts. This gives

$$
\begin{aligned}
& -\sum_{i<n} \left[p \iint_{\mathbb{R}^n_+} \partial_n \left(|v_k|^{p-2} v_k v_i \partial_k \left(\frac{a_{ni}}{a_{nn}} \right) \xi \right) t \, dX + \iint_{\mathbb{R}^n_+} \partial_n \left(\partial_i (|v_k|^p) \frac{a_{ni}}{a_{nn}} \xi \right) t \, dX \right] \\
= & -\sum_{i<n} \left[p \iint_{\mathbb{R}^n_+} \partial_n (|v_k|^{p-2} v_k) v_i \partial_k \left(\frac{a_{ni}}{a_{nn}} \right) \xi t \, dX + p \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k \partial_n (v_i) \partial_k \left(\frac{a_{ni}}{a_{nn}} \right) \xi t \, dX \right. \\
& \left. + \iint_{\mathbb{R}^n_+} \partial_i (|v_k|^p) \partial_n \left(\frac{a_{ni}}{a_{nn}} \right) \xi t \, dX \right] \qquad (6.7) \\
& -\sum_{i<n} \left[p \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k v_i \partial_k \left(\frac{a_{ni}}{a_{nn}} \right) \xi_1 \xi_2' t \, dX + \iint_{\mathbb{R}^n_+} \partial_i (|v_k|^p) \frac{a_{ni}}{a_{nn}} \xi_1 \xi_2' t \, dX \right] \\
& -\sum_{i<n} \left[p \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k v_i \partial_n \partial_k \left(\frac{a_{ni}}{a_{nn}} \right) \xi t \, dX + \iint_{\mathbb{R}^n_+} \partial_n \partial_i (|v_k|^p) \frac{a_{ni}}{a_{nn}} \xi t \, dX \right].
\end{aligned}
$$

The last two terms we integrate by parts one more time as we switch the order of derivatives. This gives

$$
\begin{aligned}
& \sum_{i<n} \left[p \iint_{\mathbb{R}^n_+} \partial_k \left(|v_k|^{p-2} v_k v_i \xi \right) \partial_n \left(\frac{a_{ni}}{a_{nn}} \right) t \, dX + \iint_{\mathbb{R}^n_+} \partial_i \left(\frac{a_{ni}}{a_{nn}} \xi \right) \partial_n (|v_k|^p) t \, dX \right]. \\
= & \sum_{i<n} \left[p \iint_{\mathbb{R}^n_+} \partial_k (|v_k|^{p-2} v_k) v_i \partial_n \left(\frac{a_{ni}}{a_{nn}} \right) \xi t \, dX + p \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k (\partial_k v_i) \partial_n \left(\frac{a_{ni}}{a_{nn}} \right) \xi t \, dX \right. \\
& \left. + \iint_{\mathbb{R}^n_+} \partial_n (|v_k|^p) \partial_i \left(\frac{a_{ni}}{a_{nn}} \right) \xi t \, dX \right] \qquad (6.8) \\
& + \sum_{i<n} \left[p \iint_{\mathbb{R}^n_+} |v_k|^{p-2} v_k v_i \partial_n \left(\frac{a_{ni}}{a_{nn}} \right) (\partial_k \xi_1) \xi_2 t \, dX + \iint_{\mathbb{R}^n_+} \partial_n (|v_k|^p) \frac{a_{ni}}{a_{nn}} (\partial_i \xi_1) \xi_2 t \, dX \right]
\end{aligned}
$$

The first three terms on the righthand side of both (6.7) and (6.8) can be bounded from above by

$$
\begin{aligned}
(6.9)\, C &\iint_{2Q\times[0,2r]} |\nabla u|^{p-1}|\nabla^2 u||\nabla A| t\, dX \\
&\le \left(\iint_{2Q\times[0,2r]} |\nabla u|^{p-2}|\nabla^2 u|^2 t\, dX\right)^{1/2}\left(\iint_{2Q\times[0,2r]} |\nabla u|^{p}|\nabla A|^2 t\, dX\right)^{1/2} \\
&\le \left(\int_{2Q} N(\nabla u)^{p-2}\iint_{\Gamma(x)} |\nabla^2 u(X)|^2 t^{2-n}\, dX\, dx\right)^{1/2}\left(\iint_{2Q\times[0,2r]} |\nabla u|^{p}|\nabla A|^2 t\, dX\right)^{1/2} \\
&\le \left(\int_{2Q} N(\nabla u)^{p-2} S^2(\nabla u)\, dx\right)^{1/2} \varepsilon \|N(\nabla u)\|^{p/2}_{L^p(2Q)} = \varepsilon \|S(\nabla u)\|_{L^p(2Q)}\|N(\nabla u)\|^{p-1}_{L^p(2Q)}.
\end{aligned}
$$

The fourth term on righthand side of (6.7) can be estimated by

$$
\begin{aligned}
(6.10)\qquad & C\iint_{2Q\times[r,2r]} |\nabla u|^{p}|\nabla A|\tfrac{t}{r}\, dX \\
&\le \left(\iint_{2Q\times[r,2r]} |\nabla u|^{p}\tfrac{t}{r^2}\, dX\right)^{1/2}\left(\iint_{2Q\times[0,2r]} |\nabla u|^{p}|\nabla A|^2 t\, dX\right)^{1/2} \\
&\le \left(\int_{2Q} N(\nabla u)^{p}(x)\, dx\right)^{1/2} \varepsilon\|N(\nabla u)\|^{p/2}_{L^p(2Q)} = \varepsilon\|N(\nabla u)\|^{p}_{L^p(2Q)}.
\end{aligned}
$$

The fifth term on righthand side of (6.7) can be estimated by

$$
\begin{aligned}
& C\iint_{2Q\times[r,2r]} |\nabla u|^{p-1}|\nabla^2 u|\tfrac{t}{r}\, dX \\
(6.11)\qquad &\le \left(\iint_{2Q\times[r,2r]} |\nabla u|^{p}\, dX\right)^{p/(p-1)}\left(\iint_{2Q\times[0,2r]} |\nabla^2 u|^{p}\, dX\right)^{1/p} \\
&\le C(r)\iint_{K} |\nabla u|^{p}\, dX.
\end{aligned}
$$

To get the last line we used some standard elliptic estimates away from the boundary (for example, it is sufficient to generalise Caccioppoli's inequality to inhomogeneous equations via the proof in [10, p. 2]). By the Carleson condition we have $|\nabla A| \le C(r)$ there. Rest is a standard bootstrap argument using the equation $\mathbf{v} = \nabla u$ satisfies, i.e., $L\mathbf{v} = \mathrm{div}((\nabla A)\mathbf{v})$ eventually yielding L^p bounds on $\nabla \mathbf{v}$ in K.

We denote the co-normal derivative of u by $H = \sum_i a_{ni}\partial_i u = \sum_i a_{ni} v_i$ and write the first term of (6.5) as

$$
\begin{aligned}
-p\iint_{\mathbb{R}^n_+}|v_k|^{p-2}v_k\partial_k\left(\frac{H}{a_{nn}}\right)\xi\,dX &= -p\iint_{\mathbb{R}^n_+}|v_k|^{p-2}v_k\partial_k\left(\frac{H}{a_{nn}}\right)\xi(\partial_n t)\,dX\\
= \;& p\iint_{\mathbb{R}^n_+}\partial_n(|v_k|^{p-2}v_k)\partial_k\left(\frac{H}{a_{nn}}\right)\xi t\,dX + p\iint_{\mathbb{R}^n_+}|v_k|^{p-2}v_k\partial_k\left(\frac{H}{a_{nn}}\right)\xi_1\xi_2' t\,dX\\
&+p\iint_{\mathbb{R}^n_+}|v_k|^{p-2}v_k\partial_n\partial_k\left(\frac{H}{a_{nn}}\right)\xi t\,dX,
\end{aligned}
\tag{6.12}
$$

where the last term further yields:

$$
-p\iint_{\mathbb{R}^n_+}\partial_k(|v_k|^{p-2}v_k)\partial_n\left(\frac{H}{a_{nn}}\right)\xi t\,dX - p\iint_{\mathbb{R}^n_+}|v_k|^{p-2}v_k\partial_n\left(\frac{H}{a_{nn}}\right)(\partial_k\xi_1)\xi_2 t\,dX. \tag{6.13}
$$

If the derivative in the first two terms on the righthand side of (6.12) and (6.13) falls on the coefficients of the matrix A we obtain terms we have already bounded above (see (6.9) and (6.10)). If the derivative falls on H the first term on the righthand side of both (6.12) and (6.13) is bounded by

$$
\begin{aligned}
& C\iint_{\mathbb{R}^n_+}|v_k|^{p-2}|\nabla v_k||\nabla H|\xi t\,dX\\
\le\;& C\left(\iint_{\mathbb{R}^n_+}|v_k|^{p-2}|\nabla v_k|^2\xi t\,dX\right)^{1/2}\left(\iint_{\mathbb{R}^n_+}|v_k|^{p-2}|\nabla H|^2\xi t\,dX\right)^{1/2}\\
\le\;& C\left(\int_{2Q}N^{p-2}(v_k)(x)\int_{\Gamma(x)}|\nabla v_k(X)|^2t^{2-n}dX\,dx\right)^{1/2}\left(\iint_{\mathbb{R}^n_+}|\nabla_T u|^{p-2}|\nabla H|^2\xi t\,dX\right)^{1/2}\\
=\;& C\left(\int_{2Q}N^{p-2}(v_k)(x)S^2(v_k)(x)\,dx\right)^{1/2}\left(\iint_{\mathbb{R}^n_+}|\nabla_T u|^{p-2}|\nabla H|^2\xi t\,dX\right)^{1/2}\\
=\;& C\|N(v_k)\|^{p/2-1}_{L^p(2Q)}\|S(v_k)\|_{L^p(2Q)}\left(\iint_{\mathbb{R}^n_+}|\nabla_T u|^{p-2}|\nabla H|^2\xi t\,dX\right)^{1/2}.
\end{aligned}
$$

If the derivative falls on H in the second term of (6.12), we get terms of the same form as (6.10) and (6.11).

It follows that for all $k\le n-1$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}}v_k(x,0)^p\xi_1(x)dx\\
\le\;& C\|N(v_k)\|^{p/2-1}_{L^p(2Q)}\|S(v_k)\|_{L^p(2Q)}\left(\iint_{\mathbb{R}^n_+}|\nabla_T u|^{p-2}|\nabla H|^2\xi t\,dX\right)^{1/2}\\
&+\varepsilon\|S(\nabla u)\|_{L^p(2Q)}\|N(\nabla u)\|^{p-1}_{L^p(2Q)}+\varepsilon\|N(\nabla u)\|^p_{L^p(2Q)}+C(r)\iint_K|\nabla u|^p\,dX\\
&+E.
\end{aligned}
\tag{6.14}
$$

Here E denotes remainder terms; these are the last two terms of (6.8) and the last term of (6.13) when the derivative falls on H. We now sum (6.14) over all $k \leq n-1$ and also sum over all coordinate patches Q_ℓ. We notice that the error terms E with complete cancel out as $\sum_\ell(\partial_k\phi_\ell) = 0$ where $(\phi_\ell)_\ell$ is the partition of unity we considered above. This yields a global estimate

$$\int_{\partial\Omega} |\nabla_T u|^p dx \leq C\|N(v_k)\|^{p/2-1}_{L^p(\partial\Omega)}\|S(v_k)\|_{L^p(\partial\Omega)} \left(\iint_{\Omega_{2r}} |\nabla_T u|^{p-2}|\nabla H|^2\delta(X)\,dX\right)^{1/2}$$
$$+\varepsilon\|S(\nabla u)\|_{L^p(\partial\Omega)}\|N(\nabla u)\|^{p-1}_{L^p(\partial\Omega)} + \varepsilon\|N(\nabla u)\|^p_{L^p(\partial\Omega)} + C(r)\iint_{\Omega\setminus\Omega_{r/2}} |\nabla u|^p\,dX.$$

From this, by (6.2) and using Lemma 4.3, we get that for all sufficiently small $\varepsilon > 0$ the desired estimate (6.1) holds. □

Lemma 6.2. *Let $p \geq 2$ be an integer, k be an integer such that $0 \leq k \leq p-2$ and u be a solution to $Lu = \mathrm{div}A\nabla u = 0$, where L is an elliptic differential operator defined on a smooth domain Ω with bounded coefficients which are such that (4.17) is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon > 0$ such that if $C(r_0) < \varepsilon$ then for some constant $K = K(\Omega, \Lambda, n, \varepsilon, m, k) > 0$*

$$\begin{aligned}(6.15)\quad &\iint_{\Omega_r} |\nabla_T u|^{p-k-2}|H|^k|\nabla H|^2\delta(X)\,dX\\ &\leq (p-k-2)K\iint_{\Omega_{2r}} |\nabla_T u|^{p-k-3}|H|^{k+1}|\nabla H|^2\delta(X)\,dX + C(r)\iint_{\Omega\setminus\Omega_r} |\nabla u|^p\,dX\\ &\quad +K\int_{\partial\Omega} |H|^p\,dx.\end{aligned}$$

Proof. We will establish (6.15) by induction on k. If $k = 0$ by Lemma 6.1 we have:

$$\int_{\partial\Omega} N^p(\nabla u)\,dx \leq K\iint_{\Omega_r} |\nabla_T u|^{p-2}|\nabla H|^2\delta(X)\,dX + C(r)\iint_{\Omega\setminus\Omega_{r/2}} |\nabla u|^p\,dX. \tag{6.16}$$

For $k > 0$ we use (6.1) and the induction assumption (6.15) for all indices $0, 1, \ldots, k-1$. This gives

$$\begin{aligned}\int_{\partial\Omega} N^p(\nabla u)\,dx &\leq K\iint_{\Omega_r} |\nabla_T u|^{p-k-2}|H|^k|\nabla H|^2\delta(X)\,dX + C(r)\iint_{\Omega\setminus\Omega_{r/2}} |\nabla u|^p\,dX\\ &\quad +K\int_{\partial\Omega} |H|^p\,d\sigma. \qquad (6.17)\end{aligned}$$

Here $K = K(k)$ and (6.17) holds for all sufficiently small $\varepsilon > 0$. From this, the inequality

$$
\begin{aligned}
&\int_{\partial\Omega} N^p(\nabla u)\,dx + K \iint_{\Omega_r} |\nabla_T u|^{p-k-2}|H|^k|\nabla H|^2\delta(X)\,dX \\
\leq\ & 2K \iint_{\Omega_r} |\nabla_T u|^{p-k-2}|H|^k|\nabla H|^2\delta(X)\,dX \\
&+2C(r)\iint_{\Omega\setminus\Omega_{r/2}} |\nabla u|^p\,dX + 2K\int_{\partial\Omega}|H|^p\,d\sigma
\end{aligned} \tag{6.18}
$$

holds when $k=0$ without any further assumptions, and when $k>0$ under the induction hypotheses (6.15) for indices $0,1,\dots,k-1$. Let us choose a cutoff function ξ as in (6.3). To control

$$\iint_{\Omega_r} |\nabla_T u|^{p-k-2}|H|^k|\nabla H|^2\delta(X)\,dX$$

it suffices to control

$$\iint_{\mathbb{R}^n_+} |\nabla_T u|^{p-k-2}|H|^k b_{ij}(\partial_i H)(\partial_j H)\xi t\,dX = I$$

for some matrix B satisfying the ellipticity condition to be specified later.

We integrate this by parts. This gives

$$
\begin{aligned}
I\ =\ & -\frac{1}{k+1}\iint_{\mathbb{R}^n_+} |\nabla_T u|^{p-k-2}|H|^k H\partial_i(b_{ij}\partial_j H)\xi t\,dX \\
&-\frac{1}{k+1}\iint_{\mathbb{R}^n_+} |\nabla_T u|^{p-k-2}|H|^k H b_{nj}(\partial_j H)\xi\,dX \\
&-\frac{1}{k+1}\iint_{\mathbb{R}^n_+} |\nabla_T u|^{p-k-2}|H|^k H b_{nj}(\partial_j H)(\partial_i\xi) t\,dX \\
&-\frac{p-k-2}{k+1}\iint_{\mathbb{R}^n_+} |\nabla_T u|^{p-k-4}(\nabla_T u\cdot\partial_i(\nabla_T u))|H|^k H b_{nj}(\partial_j H)\xi t\,dX.
\end{aligned} \tag{6.19}
$$

The second term only appears in (6.19) if $i=n$ as the function t obviously only depends on the variable $x_n=t$. We first deal with the third term of (6.19) when $i=n$. As $|\xi_2'|\leq 2/r$ and $\xi_2'=0$ on $[0,r]$ we have that this term is bounded by

$$\int_Q\int_r^{2r} |\nabla_T u|^{p-k-2}|H|^{k+1}|\nabla H|\tfrac{t}{r}\,dX \leq \varepsilon\int_{2Q} N^p_{2r}(\nabla u)\,dx + C(r)\iint_{\Omega\setminus\Omega_r}|\nabla u|^p\,dX, \tag{6.20}$$

since this term is of same type as (6.10) and (6.11) it can be estimated as before. Now for the terms with $i<n$ in the third term of (6.19) we observe that they will cancel when we sum over the index ℓ in the partition of unity introduced via the cutoff function ξ from (6.3). We choose the matrix B so that $b_{nn}=1$. Then the second term

of (6.19) if $j = n$ looks like

$$
\begin{aligned}
& -\frac{1}{(k+1)(k+2)} \iint_{\mathbb{R}^n_+} |\nabla_T u|^{p-k-2} (\partial_n |H|^{k+2}) \xi \, dX \\
= \quad & -\frac{1}{(k+1)(k+2)} \iint_{\mathbb{R}^n_+} \partial_n (|\nabla_T u|^{p-k-2} |H|^{k+2} \xi) \, dX \\
& + \frac{1}{(k+1)(k+2)} \iint_{\mathbb{R}^n_+} \partial_n (|\nabla_T u|^{p-k-2}) |H|^{k+2} \xi \, dX \\
& + \frac{1}{(k+1)(k+2)} \iint_{\mathbb{R}^n_+} |\nabla_T u|^{p-k-2} |H|^{k+2} \xi' \, dX.
\end{aligned} \tag{6.21}
$$

Here the last term again can be estimated by a solid integral $C(r) \iint_{\Omega \setminus \Omega_r} |\nabla u|^p \, dX$ in the interior of the domain. The first term is equal to a boundary integral

$$
\frac{1}{(k+1)(k+2)} \int_{\partial\Omega} |\nabla_T u|^{p-k-2} |H|^{k+2} \, dX \leq \eta \|\nabla_T u\|^p_{L^p(\partial\Omega)} + C(\eta) \|H\|^p_{L^p(\partial\Omega)},
$$

for $\eta > 0$ arbitrary small. Note that

$$
\eta \|\nabla_T u\|^p_{L^p(\partial\Omega)} \leq \eta \|N(\nabla u)\|^p_{L^p(\partial\Omega)}.
$$

We choose $\eta > 0$ so small that we can hide the term $\eta \|N(\nabla u)\|^p_{L^p(\partial\Omega)}$ on lefthand side of (6.18).

It remains to deal with the second term of (6.21). We differentiate and change the order of derivatives ∂_n and ∇_T:

$$
\frac{p-k-2}{(k+1)(k+2)} \iint_{\Omega_{2r}} |\nabla_T u|^{p-k-4} (\nabla_T u \cdot \nabla_T \partial_n u) |H|^{k+2} \xi \, dX. \tag{6.22}
$$

We reintroduce the co-normal derivative H as $\partial_n u = \frac{H}{a_{nn}} - \sum_{j<n} \frac{a_{nj}}{a_{nn}} v_j$. We also insert a term $(\partial_n t) = 1$ into both integrals. Then we integrate by parts again in the ∂_n derivative. Whenever exactly one derivative falls on the coefficients (either a_{nn} or $\frac{a_{nj}}{a_{nn}}$) those terms are bounded by

$$
\iint_{2Q \times [0,2r]} |\nabla A| |\nabla u|^{p-1} |\nabla^2 u| t \, dX \tag{6.23}
$$

which is the term of type (6.9) and has therefore a bound of type $\varepsilon \|S(\nabla u)\|_{L^p(2Q)} \|N(\nabla u)\|^{p-1}_{L^p(2Q)}$, with ε depending on the Carleson norm. For sufficiently small ε, thanks to Lemma 4.3, this can be hidden on the lefthand side of (6.18).

If both ∂_n and ∇_T derivative fall on coefficients, there are two possibilities. The first possibility is that they fall on the same coefficient and so then we do a further integration by parts in ∇_T moving this derivative on other terms. This again will yield term of type (6.23). The second possibility is that they fall on separate coefficients and so take the form (4.16), which can be estimated appropriately with the help of Lemma 4.3. We obtain another error term when ∂_n falls on ξ, however in that case we get a term of type (6.20) we handled before. Let us deal with the term when both derivatives fall on H. In that case we have

$$-\frac{p-k-2}{(k+1)(k+2)}\iint_{\Omega_{2r}}\frac{1}{a_{nn}}|\nabla_T u|^{p-k-4}(\nabla_T u\cdot\nabla_T\partial_n H)|H|^{k+2}\xi t\,dX. \tag{6.24}$$

We move the ∇_T derivative off $\partial_n H$. We can get a term of type (6.23) and two terms that can be dominated by

$$\begin{aligned} &C(p-k-2)\iint_{\Omega_{2r}}|\nabla_T u|^{p-k-4}|\nabla(\nabla_T u)||\nabla H||H|^{k+2}t\,dX \\ +\;&C(p-k-2)\iint_{\Omega_{2r}}|\nabla_T u|^{p-k-3}|\nabla H|^2|H|^{k+1}t\,dX. \end{aligned}\tag{6.25}$$

Also, when ∇_T lands on ξ we get error terms which will cancel when we sum over coordinate patches. Observe also that the last term of (6.19) can be controlled by

$$C(p-k-2)\iint_{\Omega_{2r}}|\nabla_T u|^{p-k-3}|\nabla(\nabla_T u)||\nabla H||H|^{k+1}t\,dX \tag{6.26}$$

We now deal with the terms arising from $-\sum_{j<n}\frac{a_{nj}}{a_{nn}}v_j$. Here we write

$$\nabla_T\left(\sum_{j<n}\frac{a_{nj}}{a_{nn}}v_j\right)=\sum_{j<n}\nabla_T\left(\frac{a_{nj}}{a_{nn}}\right)\partial_j u+\sum_{j<n}\frac{a_{nj}}{a_{nn}}\partial_j(\nabla_T u).$$

The contribution of the first term here, when substituted in (6.22), can be dealt with by again introducing the factor $\partial_n t$ and integrating by parts. When ∂_n lands on $\nabla_T(a_{nj}/a_{nn})$, we can move the tangential derivates off by again integrating by parts. All this yields terms of the form (4.16) (with exponent p instead of 2) and (6.23), which can be controlled appropriately. Substituting the second term in (6.22) yields

$$\frac{1}{(k+1)(k+2)}\sum_{j<n}\iint_{\Omega_{2r}}\frac{a_{nj}}{a_{nn}}\partial_j(|\nabla_T u|^{p-k-2})|H|^{k+2}\xi(\partial_n t)\,dX. \tag{6.27}$$

Moving ∂_n across using integration by parts and if necessary moving ∂_j we obtain terms either bounded by (4.16) (with exponent p instead of 2), (6.23), (6.25) or (6.26). Thus the analysis of the second term of (6.19) for $j=n$ reduces to controlling (6.25) and (6.26), a task which we will postpone for now. When $j<n$ in the second term of (6.19) we again introduce $(\partial_n t)$. This gives

$$-\iint_{\Omega_{2r}}|\nabla_T u|^{p-k-2}b_{nj}\partial_j(|H|^{k+2})\xi(\partial_n t)\,dX$$

We integrate by parts. When ∂_n falls on $|\nabla_T u|^{p-k-2}$ we can dominate such a term by (6.26), when ∂_n falls on b_{nj} we obtain a terms of type (4.16) (with exponent p instead of 2) and (6.23) and, provided we choose matrix B so that coefficients of B also satisfy the Carleson condition. If ∂_n hits ξ we get terms which can be bounded by (6.10) and (6.11). Finally the remaining term is

$$\iint_{\Omega_{2r}}|\nabla_T u|^{p-k-2}b_{nj}\partial_j\partial_n(|H|^{k+2})\xi t\,dX.$$

We integrate by parts again in ∂_j giving us terms of type (4.16) (with exponent p instead of 2), (6.23), (6.25) and (6.26). The only remaining terms we have not yet bounded are the first term of (6.19), (6.25) and (6.26). The second term of (6.25) is already of desired form (see righthand side of (6.15)). By the Cauchy-Schwarz inequality, the first term of (6.25) can be bounded by

$$\begin{aligned} C(p-k-2) &\left(\iint_{\Omega_{2r}} |\nabla_T u|^{p-k-3}|\nabla H|^2|H|^{k+1}t\,dX\right)^{1/2}\times \\ &\left(\iint_{\Omega_{2r}} |\nabla_T u|^{p-k-5}|\nabla(\nabla_T u)|^2|H|^{k+3}t\,dX\right)^{1/2} \\ \le C(p-k-2) &\left(\iint_{\Omega_{2r}} |\nabla_T u|^{p-k-3}|\nabla H|^2|H|^{k+1}t\,dX\right)^{1/2}\|N(\nabla u)\|^{p/2-1}_{L^p(\partial\Omega)}\|S(\nabla u)\|_{L^p(\partial\Omega)}. \end{aligned}$$

The last line can be further bounded by

$$\eta\|N(\nabla u)\|^p_{L^p(\partial\Omega)} + C(\eta)(p-k-2)^2\iint_{\Omega_{2r}} |\nabla_T u|^{p-k-3}|\nabla H|^2|H|^{k+1}t\,dX,$$

for $\eta>0$ arbitrary small. Hence as before we can hide $\eta\|N(\nabla u)\|^p_{L^p(\partial\Omega)}$ on the lefthand side of (6.18). Term (6.26) can be dealt with in a very similar fashion. We summarize what we have so far. By (6.18) and all estimates above we have

$$\begin{aligned} &\alpha\int_{\partial\Omega} N^p(\nabla u)\,d\sigma + \iint_{\Omega_r}|\nabla_T u|^{p-k-2}|H|^k|\nabla H|^2\delta(X)\,dX \\ \le\ & K(p-k-2)\iint_{\Omega_r}|\nabla_T u|^{p-k-3}|H|^{k+1}|\nabla H|^2\delta(X)\,dX \\ &+2C(r)\iint_{\Omega\setminus\Omega_{r/2}}|\nabla u|^p\,dX + K\int_{\partial\Omega}|H|^p\,d\sigma \\ &-\frac{K}{k+1}\iint_{\Omega_{2r}}|\nabla_T u|^{p-k-2}|H|^kH(\widetilde{L}H)\xi t\,dX. \end{aligned}\tag{6.28}$$

Here $\widetilde{L}H=\operatorname{div}(B\nabla H)$ and $\alpha>0$. The precise value of α depends on choice of $\eta>0$ above and $\varepsilon>0$. Clearly, (6.28) is the desired estimate (6.15) modulo the last extra term we shall consider now.

As above we use the summation convention, we only write the sum explicitly whenever we do not sum over all indices. For $\widetilde{L}H$ we have

$$\widetilde{L}H=\partial_i(b_{ij}\partial_jH)=\sum_{j<n}\partial_i(b_{ij}\partial_j(a_{nk}\partial_ku))+\partial_i(b_{in}\partial_n(a_{nk}\partial_ku)).$$

Since $Lu=0$ we have that $\partial_n(a_{nk}\partial_ku)=-\sum_{j<n}\partial_j(a_{jk}\partial_ku)$. Hence

$$\widetilde{L}H=\partial_i(b_{ij}\partial_jH)=\sum_{j<n}[\partial_i(b_{ij}\partial_j(a_{nk}\partial_ku))-\partial_i(b_{in}\partial_j(a_{jk}\partial_ku))].$$

We also swap the role of i and k in the second term. From this

$$\widetilde{L}H=\partial_i(b_{ij}\partial_jH)=\sum_{j<n}[\partial_i(b_{ij}\partial_j(a_{nk}\partial_ku))-\partial_k(b_{kn}\partial_j(a_{ji}\partial_iu))].\tag{6.29}$$

We choose $b_{ij} = a_{ji}/a_{nn}$. Notice that this guarantees that $b_{nn} = 1$ and that terms in (6.29) where three derivatives fall on u vanish as these are the terms:

$$\sum_{j<n}[b_{ij}a_{nk}(\partial_i\partial_j\partial_k u) - b_{kn}a_{ji}(\partial_i\partial_j\partial_k u)] = \sum_{j<n} a_{nn}^{-1}(a_{ji}a_{nk} - a_{nk}a_{ji})\partial_i\partial_j\partial_k u = 0. \tag{6.30}$$

We now place (6.29) into last term of (6.28). Given (6.30) some of the remaining terms are

$$\iint_{\Omega_{2r}} \sum_{j<n}[b_{ij}(\partial_i\partial_j a_{ij})(\partial_k u) - b_{kn}(\partial_k\partial_j a_{ji})(\partial_i u)]|\nabla_T u|^{p-k-2}|H|^k H\xi t\, dX \tag{6.31}$$

and the rest can be bounded by

$$\iint_{\Omega_{2r}} |\nabla u|^{p-1}[|\nabla u||\nabla A||\nabla B| + |\nabla^2 u||\nabla A||B| + |\nabla^2 u||\nabla B||A|]t\, dX. \tag{6.32}$$

The terms in (6.31) have two derivatives on coefficients a_{ij} however one is ∂_j and $j<n$. We therefore integrate by parts in ∂_j. This yields additional terms, but all are of the form (6.32). However, by an estimate similar to (6.23) we get that all the terms of (6.32) are smaller than $\varepsilon \int_{\partial\Omega} N^p_{3r}(\nabla u)\, d\sigma$, with ε depending on the Carleson norm of the coefficients of our operator. Hence for sufficiently small ε this term can be hidden in (6.28) within the term $\alpha \int_{\partial\Omega} N^p(\nabla u)\, dx$. This yields the desired estimate (6.15). $\square$

7. The $(N)_p$ Neumann Problem

Theorem 7.1. *Let $p \geq 2$ be an integer and let $\Omega \subset M$ be a Lipschitz domain with Lipschitz norm L on a smooth Riemannian manifold M and $Lu = div(A\nabla u)$ be an elliptic differential operator defined on Ω with ellipticity constant Λ and coefficients such that*

$$\sup\{\delta(X)|\nabla a_{ij}(Y)|^2 \,:\, Y \in B_{\delta(X)/2}(X)\} \tag{7.1}$$

is the density of a Carleson measure on all Carleson boxes of size at most r_0 with norm $C(r_0)$. Then there exists $\varepsilon = \varepsilon(\Lambda, n, p) > 0$ such that if $\max\{L, C(r_0)\} < \varepsilon$ then the Neumann problem

$$\begin{aligned} &Lu = 0, && \text{in } \Omega,\\ &A\nabla u\cdot\nu = f, && \text{on } \partial\Omega,\\ &N(\nabla u) \in L^p(\partial\Omega), && \end{aligned}$$

is solvable for all f in $L^p(\partial\Omega) < \infty$ such that $\int_{\partial\Omega} f d\sigma = 0$. Moreover, there exists a constant $C = C(\Lambda, n, a, p) > 0$ such that

$$\|N(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}. \tag{7.2}$$

Proof. For any f in the Besov space $B^{2,2}_{-1/2}(\partial\Omega)$ such that $\int_{\partial\Omega} f d\sigma = 0$ the exists a unique (up to a constant) $H^2_1(\Omega)$ solution by the Lax-Milgram theorem. Observe that our $f \in L^p(\partial\Omega) \subset B^{2,2}_{-1/2}(\partial\Omega)$ $(p \geq 2)$ so it only remains to establish the estimate (7.2).

Consider $\varepsilon > 0$ and take $C(r_0) < \varepsilon$. To keep matters simple let us first consider the case when $\partial\Omega$ is smooth. In this case Lemmas 6.1 and 6.2 apply directly. If follows that for all small r and $\varepsilon > 0$

$$\int_{\partial\Omega} N^p(\nabla u)\, d\sigma \leq K \int_{\partial\Omega} |A\nabla u \cdot \nu|^p\, d\sigma + C(r)\|\nabla u\|^p_{L^p(\Omega\setminus\Omega_r)}. \tag{7.3}$$

Here we are using Lemma 6.2 for $k = 0, 1, 2, \dots, p-2$ while observing that for the integer $k = p-2$, the first term on the righthand side of (6.15) is zero. As $A\nabla u \cdot \nu = f$ we have for non-tangential maximal function

$$\int_{\partial\Omega} N^p(\nabla u)\, d\sigma \leq K \int_{\partial\Omega} |f|^p\, d\sigma + C(r)\|\nabla u\|^p_{L^p(\Omega\setminus\Omega_r)}. \tag{7.4}$$

We also observe that we have a pointwise estimates on $\nabla u(X)$ for all X away from the boundary. There, by the Carleson condition, we have $|\nabla A| \leq C(r)$. The rest is a standard bootstrap argument using the equation $\mathbf{v} = \nabla u$ satisfies, i.e., $L\mathbf{v} = \operatorname{div}((\nabla A)\mathbf{v})$ eventually yielding pointwise bound on $|\nabla u|$ for $\{X \in \partial\Omega;\ \operatorname{dist}(X, \partial\Omega) \geq r\}$.

This yields

$$\|\nabla u\|^p_{L^p(\Omega\setminus\Omega_r)} \leq C(p)\|u\|^p_{H^2_1(\Omega)} \leq C(p)\|f\|^p_{B^{2,2}_{-1/2}(\partial\Omega)}. \tag{7.5}$$

Finally, combining (7.4) and (7.5) we obtain the desired estimate (7.2).

Now we turn to the more general case, when Ω has a Lipschitz boundary with sufficiently small Lipschitz constant L. This case also includes the C^1 boundary as in this case L can be taken arbitrary small.

The crucial point is that the proofs of Lemmas 4.7, 6.1 and 6.2 in the smooth case are based on local estimates near boundary $\partial\Omega$. This means we can reduce the matter to a local situation working on the subset of upper half-space via the map (4.10) as we did above. From this the claim follows on domains with small Lipschitz constant, as well as on domains whose boundaries are given locally by functions with gradient in VMO. $\square$

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